Theorem 20.3 Let $p(X) \in K[X]$ be irreducible of degree d and $A \in M_{n\times n}(K)$ be such that

$$M_A \cong K[X]/(p(X)^k)$$

for some $k > 0$. Then A is similar to a matrix of the form

$$J_{k,p(X)} = \begin{pmatrix} C_{p(X)} & & & O \\ N & C_{p(X)} & & \\ & N & \ddots & \\ O & & & C_{p(X)} \end{pmatrix} \in M_{dk\times dk}(K) \quad \begin{pmatrix} k \text{ copies} \\ \text{of } C_{p(X)} \end{pmatrix}$$

where $N = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix}$. Such a matrix is called a <u>generalized Jordan block</u>.

<u>Proof.</u> Omitted.

Theorem 20.4 Let $A \in M_{n\times n}(K[X])$. Then there exist monic irreducible polynomials $p_1(X), \dots, p_s(X) \in K[X]$ and integers $k_{1,1}, \dots, k_{1,r_1}, k_{2,1}, \dots, k_{2,r_2}, \dots, k_{s,1}, \dots, k_{s,r_s}$ such that A is similar to a matrix of the form

$$J = \begin{pmatrix} J_{k_{1,1},p_1(X)} & & & & & & \\ & \ddots & & & & & \\ & & J_{k_{1,r_1},p_1(X)} & & & & \\ & & & J_{k_{2,1},p_2(X)} & & & \\ & & & & \ddots & & \\ & & & & & J_{k_{2,r_2},p_2(X)} & \\ & & & & & & \ddots \\ & & & & & & & J_{k_{s,1},p_s(X)} \\ & & & & & & & & \ddots \\ & & & & & & & & & J_{k_{s,r_s},p_s(X)} \end{pmatrix},$$

called the <u>Jordan canonical form</u> of A, $p_1(X), \dots, p_s(X)$ are called the <u>elementary divisors</u> of A, and J is unique up to reordering of diagonal blocks

<u>Proof.</u> Follows by (*), (**) and Theorem 20.3 (we may always assume the polynomials to be monic by multiplying by the inverse of the coefficient of the highest degree term. $\square$

Now we specialize to the situation where $K = \mathbb{C}$. Then every polynomial of degree d has, up to multiplicity, d roots and so the only monic irreducible polynomials in $\mathbb{C}[X]$ are

$$p_\lambda(X) = X - \lambda, \quad \lambda \in \mathbb{C}.$$

Then

$$J_{k, p_\lambda(X)} = \begin{pmatrix} \lambda & & & 0 \\ 1 & \lambda & & \\ & 1 & \ddots & \\ 0 & & & \lambda \end{pmatrix} \in M_{k \times k}(\mathbb{C}).$$

Moreover, for $\lambda_1, \dots, \lambda_s$ pairwise different we have

$$\frac{K[X]}{((X-\lambda_1)^{k_1} \cdots (X-\lambda_s)^{k_s})} \cong \frac{K[X]}{((X-\lambda_1)^{k_1})} \oplus \dots \oplus \frac{K[X]}{((X-\lambda_s)^{k_s})}.$$

Hence given the rational canonical form of A

$$A \sim \begin{pmatrix} C_{p_1(X)} & & 0 \\ & \ddots & \\ 0 & & C_{p_t(X)} \end{pmatrix},$$

we may split all of the polynomials $p_1(X), \dots, p_t(X)$ into factors $(X - \lambda_i)^{k_i}$ to obtain the Jordan normal form of A.

<u>Example 20.5.</u> (Exam December 2011, Problem 1(c)). We have computed the rational canonical form of

$$A = \begin{pmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 0 & -4 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ 0 & 0 & 0 & -4 \\ 0 & 0 & 1 & 4 \end{pmatrix} \sim \left(\begin{array}{c|c} C_{(X-2)^2} & O \\ \hline O & C_{(X-2)^2} \end{array}\right)$$

in Example 19.8. Hence the elementary divisors are $(X-2)^2$, $(X-2)^2$ and $X-2$ is irreducible in $\mathbb{Q}[X]$. Then

$$A \sim \left(\begin{array}{c|cc} J_{2,(X-2)} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \hline \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & J_{2,(X-2)} \end{array}\right) = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

is the Jordan canonical form of A.