20. Jordan canonical form (Chapter 21.5)

K - field
$R = K[X]$ (unless stated otherwise)

<u>Motivation</u>: Let $A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \in \mathbb{C}[X]$. To compute its rational canonical form we have

$$\begin{pmatrix} 3-X & 0 \\ 0 & 2-X \end{pmatrix} \xrightarrow{R_1 \to R_1 + R_2} \begin{pmatrix} 3-X & 2-X \\ 0 & 2-X \end{pmatrix} \xrightarrow{C_1 \to C_1 - C_2}$$

$$\begin{pmatrix} 1 & 2-X \\ X-2 & 2-X \end{pmatrix} \xrightarrow{C_2 \to C_2 - (2-X)C_1} \begin{pmatrix} 1 & 0 \\ 2-X & X^2-5X+6 \end{pmatrix} \xrightarrow{R_2 \to R_2 - (2-X)R_1}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & X^2-5X+6 \end{pmatrix}.$$

We could also compute this by noticing that
$$c_A(X) = \det(A - XI_2) = \begin{vmatrix} 3-X & 0 \\ 0 & 2-X \end{vmatrix} = (3-X)(2-X)$$
and so the only possibility for the Smith normal form of $A - XI_n$ is
$$\begin{pmatrix} 1 & 0 \\ 0 & (3-X)(2-X) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & X^2-5X+6 \end{pmatrix}.$$
Then the rational canonical form of A is
$$C_{6-5X+X^6} = \begin{pmatrix} 0 & -6 \\ 1 & 5 \end{pmatrix}$$
which is "more complicated" than A. We are going to search for another representative of the similarity class.

<u>Definition 20.1</u> Let R be a PID and $p \in R$ be a prime = irreducible element. The <u>p-primary</u> part of an R-module M is the submodule
$$M_p = \{ m \in M \mid p^k m = 0 \text{ for some } k > 0 \},$$

and M is called <u>p-primary</u> if $M = M_p$.

<u>Theorem 20.2</u> Let R be a PID and let M be a finitely generated R-module. There exists a finite set $\{p_1, \dots, p_k\}$ of irreducible elements in R such that $\operatorname{Tor} M = \bigoplus_{i=1}^{k} M_{p_i}$.

<u>Proof.</u> For any irreducible element p in R we have $M_p \subseteq \operatorname{Tor} M$ by definition. Hence $\sum_{p \text{ irreducible}} M_p \subseteq \operatorname{Tor} M$. Now let $m \in \operatorname{Tor} M$. Then there exist $r \in R \setminus \{0\}$ such that

$$rm = 0.$$

Since R is a PID, it is also a UFD and so we may assume

$$r = p_1^{k_1} \cdots p_n^{k_n}, \quad \text{for irreducible } p_i \in R.$$

For $1 \le j \le n$, set $q_j = \prod_{i \ne j} p_i^{k_i}$. Then

$$rm = 0 \Rightarrow p_1^{k_1} \cdots p_n^{k_n} m = 0 \Rightarrow p_j^{k_j}(q_j m) = 0 \Rightarrow q_j m \in M_{p_j}.$$

Since $\gcd(q_1, \dots, q_n) = 1$, there exist $a_1, \dots, a_n \in R$ such that

$$1 = a_1 q_1 + \dots + a_n q_n \qquad \text{(Problem 1 in Problem Set 5)}.$$

Then we obtain

$$m = 1 \cdot m = a_1 \underbrace{q_1 m}_{\in M_{p_1}} + \dots + a_n \underbrace{q_n m}_{\in M_{p_n}} \in M_{p_1} + \dots + M_{p_n} \subseteq \sum_{p \text{ irreducible}} M_p.$$

Hence $\operatorname{Tor} M = \sum_{p \text{ irreducible}} M_p$.

If $m \in M_p \cap M_q$ for $p, q \in R$ irreducible, then we have $p^k m = 0$ and $q^\ell m = 0$ for some $k, \ell > 0$. If $p \ne q$, then there exist $a, b \in R$ with

$$1 = a p^k + b q^\ell$$

$$\Rightarrow m = a p^k m + b q^\ell m = a0 + b0 = 0$$

and so $m = 0$. Hence

$$\sum_{p \text{ irreducible}} M_p = \bigoplus_{p \text{ irreducible}} M_p.$$

Since M is finitely generated, we have that $\operatorname{Tor} M$ is finitely generated (since R is a PID) and so there exist finitely many irreducible elements $p_1, \dots, p_k \in R$ such that $M = \bigoplus_{i=1}^{k} p_i$. $\square$

Now let $A \in M_{n\times n}(k)$. Then by Lemma 19.3 we have that $M_A = \mathrm{Tor} M_A$ and so Theorem 20.2 gives

$$M_A = \mathrm{Tor} M_A = M_{A,p_1(X)} \oplus \cdots \oplus M_{A,p_s(X)} \qquad (*)$$

for some irreducible polynomials $p_i(X) \in k[X]$, where

$$M_{A,p(X)} = \{ v \in M_A = k^n \mid p(A)^k v = 0 \text{ for some } k>0 \}$$

is the $p(X)$-primary part of M_A. Since for $v \in k^n$ we have

$$p(X) \cdot_{c_A} (Av) = p(X) \cdot_{c_A} (X \cdot_{c_A} v) = (p(X)X) \cdot_{c_A} v = X(p(X) \cdot_{c_A} v),$$

we obtain that the linear map

$$T_A : k^n \longrightarrow k^n$$
$$v \longmapsto Av$$

restricts to a linear map $T_{A_{p_i(X)}} : M_{A,p_i(X)} \longrightarrow M_{A,p_i(X)}$.
Hence by choosing appropriate basis we obtain that A is similar to a matrix of the form

$$\begin{pmatrix} A_{p_1(X)} & & & 0 \\ & A_{p_2(X)} & & \\ & & \ddots & \\ 0 & & & A_{p_s(X)} \end{pmatrix}$$

where $A_{p_i(X)}$ is the matrix of the restriction $T_{A_{p_i(X)}}$.
Then by Theorem 19.5 we obtain that for $1 \le i \le s$, there exist $k_{i,1}, \dots, k_{i,r_i}$ such that

$$M_{A_{p_i(X)}} \cong \frac{k[X]}{(p_i(X)^{k_1})} \oplus \cdots \oplus \frac{k[X]}{(p_i(X)^{k_r})}. \qquad (**)$$

Hence we need to study $\frac{k[X]}{(p(X)^k)}$ for $p(X)$ irreducible.