Example 19.8. (Exam December 2011, Problem 1(a) and 1(b))

Let
$$A = \begin{pmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \in M_{4\times 4}(\mathbb{Q}).$$

To find the rational canonical form of A, we first compute the Smith normal form of $A - XI_n$

$$\begin{pmatrix} 2-X & 0 & -1 & 0 \\ 0 & 2-X & 0 & -1 \\ 0 & 0 & 2-X & 0 \\ 0 & 0 & 0 & 2-X \end{pmatrix} \xrightarrow{C_1 \leftrightarrow C_3} \begin{pmatrix} -1 & 0 & 2-X & 0 \\ 0 & 2-X & 0 & -1 \\ 2-X & 0 & 0 & 0 \\ 0 & 0 & 0 & 2-X \end{pmatrix}$$

$$\xrightarrow{C_1 \to -C_1} \begin{pmatrix} 1 & 0 & 2-X & 0 \\ 0 & 2-X & 0 & -1 \\ X-2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2-X \end{pmatrix} \xrightarrow{C_3 \to C_3 - (2-X)C_1}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2-X & 0 & -1 \\ X-2 & 0 & (X-2)^2 & 0 \\ 0 & 0 & 0 & 2-X \end{pmatrix} \xrightarrow{R_3 \to R_3 - (X-2)R_1}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2-X & 0 & -1 \\ 0 & 0 & (X-2)^2 & 0 \\ 0 & 0 & 0 & 2-X \end{pmatrix} \xrightarrow{C_2 \leftrightarrow C_4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 2-X \\ 0 & 0 & (X-2)^2 & 0 \\ 0 & 2-X & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{R_2 \to -R_2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & X-2 \\ 0 & 0 & (X-2)^2 & 0 \\ 0 & 2-X & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{R_4 \to R_4 - (2-X)R_2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & X-2 \\ 0 & 0 & (X-2)^2 & 0 \\ 0 & 0 & 0 & (X-2)^2 \end{pmatrix}$$

$$\xrightarrow{C_4 \to C_4 - (X-2)C_2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & (X-2)^2 & 0 \\ 0 & 0 & 0 & (X-2)^2 \end{pmatrix}$$

and $1 \mid 1 \mid (X-2)^2 \mid (X-2)^2$. The monic non-units are $(X-2)^2 = X^2 - 4X + 4$ (twice). Hence the rational canonical form of A is

$$\left(\begin{array}{c|c} C_{(X-2)^2} & 0 \\ \hline 0 & C_{(X-2)^2} \end{array} \right) = \begin{pmatrix} 0 & -4 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ 0 & 0 & 0 & -4 \\ 0 & 0 & 1 & 4 \end{pmatrix}.$$