19. Rational canonical form (Chapter 21.4)

K - field

Definition 19.1 A matrix $A \in M_{n\times n}(K)$ is said to be *similar* to a matrix $B \in M_{n\times n}(K)$ if there exists an invertible matrix $P \in M_{n\times n}(K)$ such that $B = PAP^{-1}$. In this case we write $A \sim B$.

Clearly being similar is an equivalence relation in $M_{n\times n}(K)$.

Aim: obtain a "canonical" representative of the similarity class of $A \in M_{n\times n}(K)$. Our main tool is applying the structure theorem for finitely generated modules over a PID on $K[X]$.

Theorem 19.2. Let M be a finitely generated $K[X]$-module. Then there exist an integer $s \geq 0$ and unique monic polynomials $p_1(X), \dots, p_u(X) \in K[X]$ with $p_1(X) \mid p_2(X) \mid \cdots \mid p_u(X)$ such that

$$M \cong K[X]^s \oplus \frac{K[X]}{(p_1(X))} \oplus \cdots \oplus \frac{K[X]}{(p_u(X))}.$$

Proof. The required form follows by Theorem 17.6, up to the fact that $p_i(X)$ can be taken to be monic, which follows since if $p(X) = a_0 + a_1X + \cdots + a_nX^n \in K[X]$, $a_n \neq 0$, then $(p(X)) = (a_n^{-1}p(X))$. By choosing monic representatives we also ensure uniqueness. $\square$