15. Row and column rank of a matrix over a PID
(Chapters 20.1 and 20.2)

R - unital PID

Lemma 15.1. Let M be a finitely generated free R-module. Then any submodule $N \subseteq M$ is also finitely generated and free and, moreover, we have $\operatorname{rank}(N) \leq \operatorname{rank}(M)$.

Proof. Since M is finitely generated and free over a commutative ring, we have that $\operatorname{rank}(M)$ is well-defined. Let $\operatorname{rank}(M) = m$. Then $M \cong R^m$ as R-modules by Theorem 12.4. Hence it is enough to show the statement for the R-module R^m.

Let $U \subseteq R^m$ be a submodule. We use induction on $m \geq 0$. If $m=0$, then $M = 0 = R^0$. This gives $U=0$ and the statement clearly holds with $\emptyset$ as a basis. For the induction step, let

$$U' = U \cap \{(r_1, \dots, r_{m-1}, 0) \in R^m\} \cong U \cap R^{m-1}.$$

Then U' is a submodule of R^{m-1} and so by induction it has a basis $\{u_1, \dots, u_t\}$ with $t \leq m-1$. Consider the R-module homomorphism

$$\begin{aligned} \pi_m : R^m &\longrightarrow R \\ (r_1, \dots, r_m) &\longmapsto r_m. \end{aligned}$$

Then $\pi_m(U) \subseteq R$ is a submodule. If $\pi_m(U) = 0$, then $U' = U$ and we are done. Otherwise, since R is a PID, there exists $a \in R \setminus \{0\}$ such that $\pi_m(U) = Ra$. Let $u_{t+1} \in U$ be such that $\pi_m(u_{t+1}) = a \in Ra$. We claim that $\{u_1, \dots, u_t, u_{t+1}\}$ is a basis of U.

Let $u \in U$. Then $\pi_m(u) \in Ra$ and so $\pi_m(u) = ra$ for some

$r \in R$. Then

$$\pi_m(u - ru_{t+1}) = \pi_m(u) - r\,\pi_m(u_{t+1}) = ra - ra = 0$$

and so $u - ru_{t+1} \in \{(r_1, \dots, r_{m-1}, 0) \in R^m\}$. Since $u, u_{t+1} \in U$, we conclude that $u - ru_{t+1} \in U'$. Since $\{u_1, \dots, u_t\}$ generates U', there exist some $r_1, \dots, r_t \in R$ such that

$$u - ru_{t+1} = r_1 u_1 + \cdots + r_t u_t$$

$$\Rightarrow \quad u = r_1 u_1 + \cdots + r_t u_t + r u_{t+1},$$

which shows that $U = \langle u_1, \dots, u_t, u_{t+1} \rangle$.

Now suppose that

$$r_1 u_1 + \cdots + r_t u_t + r_{t+1} u_{t+1} = 0. \qquad (1)$$

Then applying π_m and since $\pi_m(u_1) = \cdots = \pi_m(u_t) = 0$, we obtain that

$$r_{t+1}\pi_m(u_{t+1}) = 0 \Rightarrow r_{t+1} a = 0 \Rightarrow r_{t+1} = 0$$

where the last implication holds because R is a PID.
Then $(*)$ becomes

$$r_1 u_1 + \cdots + r_t u_t = 0$$

and so $r_1 = \cdots = r_t = 0$ since $\{r_1, \dots, r_t\}$ is R-linearly independent. Hence $\{u_1, \dots, u_t, u_{t+1}\}$ is R-linearly independent. Therefore we have shown that U is free and finitely generated and

$$\operatorname{rank}(U) = t + 1 \le m - 1 + 1 = m,$$

as required. $\square$

Now we fix a matrix $A \in M_{m\times n}(R)$, that is

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}.$$

Write $R_i(A)$ for the i-th row of A, that is

$$R_i(A) = (a_{i1}, \dots, a_{in}).$$

Then $R_i(A) \in M_{1\times n}(R) \cong R^n$. Similarly, write C_j for the j-th column of A, that is

$$C_j(A) = \begin{pmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{pmatrix}.$$

Then $C_j(A) \in M_{m\times 1}(R) \cong R^m$.

Definition 15.2. (1) The *row module* of A, denoted $R(A)$, is the submodule of R^n generated by $\{R_1(A), \dots, R_m(A)\}$ and the *column module* of A, denoted $C(A)$, is the submodule of R^m generated by $\{C_1(A), \dots, C_n(A)\}$.
(2) The *row rank* of A is the rank of $R(A)$ and the *column rank* of A is the rank of $C(A)$.

Remark 15.3. (1) The row and column rank of A are well-defined by Lemma 15.1 since $R(A)$ and $C(A)$ are submodules of R^n and R^m, respectively.
(2) If R is a field, then $R(A)$ is the row space of A and $C(A)$ is the column space of A.
(3) Let $P \in M_{m\times m}(R)$ and $Q \in M_{n\times n}(A)$ be invertible. Similarly to the case of matrices over fields, one can show that

$$\text{row rank}(PAQ) = \text{row rank}(A)$$
$$\text{column rank}(PAQ) = \text{column rank}(A).$$