Theorem 14.6. (Wedderburn-Artin) The following are equivalent.

(1) R is a left semisimple ring.

(2) There exists division rings $D_1, \dots, D_k$ and positive integers $n_1, \dots, n_k$ such that
$$R \cong M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k).$$

Proof. (2) $\Rightarrow$ (1) Follows by Example 14.2(2).

(1) $\Rightarrow$ (2). Since ${}_RR$ is semisimple, we have that ${}_RR = \bigoplus_{i \in I} S_i$ for some family of simple left R-modules S_i, $i \in I$. Since R is left artinian by Proposition 14.5, we conclude that only finitely many of these are nonzero, say $S_1, \dots, S_m$. We group these simple modules according to whether they are isomorphic or not. Hence, after relabelling, say we have n_1 isomorphic copies of a simple module S_1, n_2 isomorphic copies of a simple module S_2, etc., up to n_k isomorphic copies of a simple module S_k. Then

$${}_RR \cong \underbrace{S_1 \oplus \cdots \oplus S_1}_{n_1 \text{ summands}} \bigoplus \underbrace{S_2 \oplus \cdots \oplus S_2}_{n_2 \text{ summands}} \bigoplus \cdots \bigoplus \underbrace{S_k \oplus \cdots \oplus S_k}_{n_k \text{ summands}}.$$

By Schur's lemma (Theorem 11.4) we have that
$$\mathrm{Hom}_R(S_i, S_j) \cong \begin{cases} 0, & i \neq j, \\ D_i, & i = j, \end{cases}$$
where D_i is a division ring. Then Theorem 14.3 gives

$$\mathrm{Hom}_R({}_RR,{}_RR) \cong \left(\begin{array}{c|c|c|c}
\underbrace{\begin{matrix} D_1 & \cdots & D_1 \\ \vdots & & \vdots \\ D_1 & \cdots & D_1 \end{matrix}}_{n_1}\left.\vphantom{\begin{matrix} D_1 \\ \vdots \\ D_1 \end{matrix}}\right\}n_1 & 0 & 0 & 0 \\ \hline
0 & \underbrace{\begin{matrix} D_2 & \cdots & D_2 \\ \vdots & & \vdots \\ D_2 & \cdots & D_2 \end{matrix}}_{n_2}\left.\vphantom{\begin{matrix} D_2 \\ \vdots \\ D_2 \end{matrix}}\right\}n_2 & 0 & 0 \\ \hline
0 & 0 & \ddots & 0 \\ \hline
0 & 0 & 0 & \begin{matrix} D_k & \cdots & D_k \\ \vdots & & \vdots \\ D_k & \cdots & D_k \end{matrix}
\end{array}\right)$$

$$\cong M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k).$$

Using Proposition 10.9 we obtain

$$R \cong \mathrm{Hom}_R({}_RR,{}_RR)^{op} \cong (M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k))^{op}$$
$$\cong M_{n_1}(D_1)^{op} \times \cdots \times M_{n_k}(D_k)^{op}$$
$$\cong M_{n_1}(D_1^{op}) \times \cdots \times M_{n_k}(D_k^{op}). \quad \text{(exercise)}$$

But if D is a division ring, then D^{op} is also a division ring and so (2) is shown. $\square$

<u>Corollary 14.7</u> The following are equivalent.

(1) R is left semisimple.

(2) R is right semisimple.

If R satisfies any of the above conditions, then we simply say that R is <u>semisimple</u>.

<u>Proof</u>. Exercise.

The decomposition in the Wedderburn–Artin theorem is unique up to reordering and isomorphism as the following shows.

Proposition 14.8. (1) Let D, D' be division rings and assume that $M_n(D) \cong M_{n'}(D')$ for some $n, n' > 0$. Then $n = n'$ and $D \cong D'$.
(2) Let $D_1, \dots, D_p$ and $D'_1, \dots, D'_q$ be division rings. Let $n_1, \dots, n_p, k_1, \dots, k_q > 0$ be integers. If
$$M_{n_1}(D_1) \times \dots \times M_{n_p}(D_p) \cong M_{k_1}(D'_1) \times \dots \times M_{k_q}(D'_q)$$
holds, then $p = q$ and there exists a bijection $\sigma \in S_p$ such that $n_i = k_{\sigma(i)}$ and $D_i \cong D'_{\sigma(i)}$.

Proof. (1) Omitted, see Theorem 13.3.8 in the book.
(2) That $p = q$ is left as an exercise. We use induction on p. If $p = 1$, then the claim follows by part (1). Now assume the claim holds for $p-1$. We have that $M_{n_p}(D_p) \cong 0 \times \dots \times 0 \times M_{n_p}(D_p)$ is a two-sided ideal of $M_{n_1}(D_1) \times \dots \times M_{n_p}(D_p)$. Therefore, it corresponds to a two-sided ideal of $M_{k_1}(D'_1) \times \dots \times M_{k_q}(D'_q)$. Two-sided ideals of $M_{k_1}(D'_1) \times \dots \times M_{k_p}(D'_p)$ are of the form $I_1 \times \dots \times I_p$ where I_i is a two sided-ideal of $M_{k_i}(D'_i)$. By Theorem 3.4(3) we have that $I_i = 0$ or $I_i = M_{k_i}(D'_i)$. Hence

$$M_{n_1}(D_1) \times \dots \times M_{n_{p-1}}(D_{p-1}) \cong \frac{M_{n_1}(D_1) \times \dots \times M_{n_p}(D_p)}{0 \times \dots \times M_{n_p}(D_p)}$$
$$\cong \frac{M_{k_1}(D'_1) \times \dots \times M_{k_p}(D'_p)}{I_1 \times \dots \times I_p}$$
$$\cong \frac{M_{k_1}(D'_1)}{I_1} \times \dots \times \frac{M_{k_p}(D'_p)}{I_p}.$$

The right-hand side is again isomorphic to a product of matrix rings over division rings, since each I_i is either 0 or $M_{k_i}(D_i')$. Hence we must have exactly $p-1$ factors in the right-hand side, since we have $p-1$ factors in the left-hand side. We conclude that there exists some j such that

$$I_1 \times \cdots \times I_p = 0 \times \cdots \times 0 \times M_{k_j}(D_j') \times 0 \times \cdots \times 0$$

and so

$$M_{n_1}(D_1) \times \cdots \times M_{n_{p-1}}(D_{p-1}) \cong M_{k_1}(D_1') \times \cdots \times M_{k_{j-1}}(D_{j-1}') \times M_{k_{j+1}}(D_{j+1}') \times \cdots \times M_{k_p}(D_p').$$

By induction hypothesis we obtain a bijection $\sigma \in S_{p-1}$ between $\{1, \ldots, p-1\}$ and $\{1, \ldots, j-1, j+1, \ldots, p\}$ such that $n_i = k_{\sigma(i)}$ and $D_i \cong D_{\sigma(i)}'$. Then we have

$$M_{n_p}(D_p) \cong \frac{M_{n_1}(D_1) \times \cdots \times M_{n_p}(D_p)}{M_{n_1}(D_1) \times \cdots \times M_{n_{p-1}}(D_{p-1}) \times 0}$$

$$\cong \frac{M_{k_1}(D_1') \times \cdots \times M_{k_p}(D_p')}{M_{k_1}(D_1') \times \cdots \times M_{k_{j-1}}(D_{j-1}') \times 0 \times M_{k_{j+1}}(D_{j+1}') \times \cdots \times M_{k_p}(D_p')}$$

$$\cong M_{k_j}(D_j')$$

which completes the proof. $\square$