13. Noetherian and artinian modules (Chapters 19.1–19.2)

R – unital ring
M – left R-module (but everything holds for right modules too)

Proposition 13.1. The following are equivalent.
(1) M is finitely generated.
(2) Whenever $N_i, i \in I$ is a family of submodules of M such that $M = \sum_{i \in I} N_i$, there exists some finite subset $I' \subseteq I$ such that $M = \sum_{i \in I'} N_i$.

Proof (1)$\Rightarrow$(2). Since M is finitely generated, there exist $m_1, \dots, m_n \in M$ such that $M = (m_1, \dots, m_n)$. Let $N_i, i \in I$ be a family of submodules of M such that $M = \sum_{i \in I} N_i$. Then $m_j \in M$ and so $m_j \in \sum_{i \in I_j} N_i$ for some finite $I_j \subseteq I$. Set $I' = I_1 \cup \dots I_j$. Then $I' \subseteq I$ is finite and

$$M = (m_1, \dots, m_n) \subseteq \left(\bigcup_{i \in I'} N_i \right) \xlongequal{\text{Theorem 9.13}} \sum_{i \in I'} N_i \subseteq \sum_{i \in I} N_i = M.$$

Hence $M = \sum_{i \in I'} N_i$.

(2)$\Rightarrow$(1). We have that $M = \sum_{m \in M} (m)$. Hence by (2) there exists a finite set $\{m_1, \dots, m_n\} \subseteq M$ such that $M = \sum_{i=1}^n (m_i)$. Then

$$M = \sum_{i=1}^n (m_i) \xlongequal{\text{Theorem 9.13}} \left(\bigcup_{i=1}^n (m_i) \right) \subseteq (m_1, \dots, m_n) = M,$$

and so $M = (m_1, \dots, m_n)$ is finitely generated. $\square$

Motivated by the above we give the following dual definition.

Definition 13.2. M is said to be finitely cogenerated if whenever $N_i, i \in I$ is a family of submodules of M such that $\{0\} = \bigcap_{i \in I} N_i$, there exists some finite subset $I' \subseteq I$ such that $\{0\} = \bigcap_{i \in I'} N_i$.

Our aim is to characterise finitely generated and cogenerated modules via chains of submodules.