NTNU
Norwegian University of Science and Technology

Department of Mathematical Sciences

Examination paper for **MA3201 Rings and Modules**

Academic contact during examination: Sverre O. Smalø
Phone: 48293975

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Problem 1

a) Find the Smith normal form of the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 \\ 1 & 3 & 9 & 27 & 81 \\ 1 & 4 & 16 & 64 & 256 \\ 1 & 5 & 25 & 125 & 625 \end{pmatrix}$$

over $\mathbb{Z}$.

Problem 2 Let $M_2(\mathbb{Z})$ be the ring of 2×2 matrices over $\mathbb{Z}$, i.e.

$$M_2(\mathbb{Z}) = \{\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \mid a_1, a_2, b_1, b_2, \in \mathbb{Z}\},$$

with the usual matrix addition and multiplication, and let $M_2(\mathbb{Z}_n)$ be the ring of 2×2 matrices over $\mathbb{Z}_n$, i.e.

$$M_2(\mathbb{Z}_n) = \{\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \mid a_1, a_2, b_1, b_2, \in \mathbb{Z}_n\},$$

with the usual matrix addition and multiplication modulo n.

a) Prove that for each $n \in \mathbb{Z} \setminus \{0\}$, the map $\Phi_n : M_2(\mathbb{Z}) \to M_2(\mathbb{Z}_n)$, given by

$$\Phi_n(\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}) = \begin{pmatrix} a_1 \mod n & a_2 \mod n \\ b_1 \mod n & b_2 \mod n \end{pmatrix},$$

is a surjective ring homomorphism and find the kernel of Φ_n.

b) Let $n = st$ with $s,\ t \in \mathbb{Z} \setminus \{0, 1, -1\}$, and prove that $\Phi_{s,t} : M_2(\mathbb{Z}) \to M_2(\mathbb{Z}_s) \times M_2(\mathbb{Z}_t)$ given by

$$\Phi_{s,t}(\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}) = (\Phi_s(\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}), \Phi_t(\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix})),$$

is a ring homomorphism and find its kernel.

c) When will $\Phi_{s,t}$, as defined in part **b)**, be surjective? You have to give a reason for your answer.

d) For which n will the ring $M_2(\mathbb{Z}_n)$ be semi-simple? You have to give a reason for your answer.

Problem 3 Let R be the subset of the ring of 2×2 matrices over $\mathbb{Z}$ given as

$$\{a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \mid a,\ b \in \mathbb{Z}\}$$

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a) Show that R is a commutative ring which is an integral domain, and find the group of units in R. Hint: Observe that the second matrix satisfy the polynomial $X^2 + X + 1$ or use determinants.

b) Find the ideal generated by the element $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and show that $R/\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} R$ is a field with 4 elements.

c) Show that the ring $R/\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} R$ is not semisimple. Hint: Try to find a nonzero nilpotent element.

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Problem 4

a) Prove that every nonzero finitely generated $\mathbb{Z}$-submodule of $\mathbb{Q}$ is a free $\mathbb{Z}$-module on one generator. Hint: Show that a submodule generated by two elements, is in fact generated by one element.

b) Prove that $\mathbb{Q}$ is not noetherian as a $\mathbb{Z}$-module.