Select Solutions from 2015 MA3201 Exam

1a Find Smith normal form of $\begin{bmatrix} 2-x & 1 & 2 \\ 0 & 1-x & 2 \\ 1 & 0 & 1-x \end{bmatrix}$ over $\mathbb{Z}/5\mathbb{Z}[x]$.

$$\begin{bmatrix} 2-x & 1 & 2 \\ 0 & 1-x & 2 \\ 1 & 0 & 1-x \end{bmatrix} \xrightarrow{R_1-(2-x)R_3} \begin{bmatrix} 0 & 1 & 2-(2-x)(1-x) \\ 0 & 1-x & 2 \\ 1 & 0 & 1-x \end{bmatrix}$$

$(2-x)(1-x) = 2-3x+x^2$

$$\xrightarrow{R_3 \leftrightarrow R_1} \begin{bmatrix} 1 & 0 & 1-x \\ 0 & 1-x & 2 \\ 0 & 1 & -x^2+3x \end{bmatrix} \xrightarrow{C_3-(1-x)C_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1-x & 2 \\ 0 & 1 & -x^2+3x \end{bmatrix}$$

$$\xrightarrow{R_2-(1-x)R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2-(1-x)(-x^2+3x) \\ 0 & 1 & -x^2+3x \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -x^2+3x \\ 0 & 0 & -x^3+4x^2-3x+2 \end{bmatrix}$$

$(1-x)(-x^2+3x) = x^3-4x^2+3x$

$$\xrightarrow{C_3-(-x^2+3x)C_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -x^3+4x^2-3x+2 \end{bmatrix}$$

$$\underset{\text{over } \mathbb{Z}/5\mathbb{Z}[x]}{=} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4x^3+4x^2+2x+2 \end{bmatrix}$$

$4 \cdot 4 = 16 \equiv 1 \bmod 5$

unit

$2 \cdot 4 = 8 \equiv 3 \bmod 5$

unit

$$\xrightarrow{4R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & x^3+x^2+3x+2 \end{bmatrix}$$

(1b) Find rational canonical form of $A = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$ over $\mathbb{Z}/5\mathbb{Z}$.

(First find invariant factors of $A - xI_3$.)

$$A - xI_3 = \begin{bmatrix} 2-x & 1 & 2 \\ 0 & 1-x & 2 \\ 1 & 0 & 1-x \end{bmatrix}$$

Smith normal form (from 1a) : $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & x^3+x^2+3x+3 \end{bmatrix}$.

non-unit invariant factor :

$x^3 + x^2 + 3x + 3$.

Companion matrix : $\begin{bmatrix} 0 & 0 & -3 \\ 1 & 0 & -3 \\ 0 & 1 & -1 \end{bmatrix}$

rational canonical form (has 1 block) :

$$\begin{bmatrix} 0 & 0 & -3 \\ 1 & 0 & -3 \\ 0 & 1 & -1 \end{bmatrix}.$$

(1c) Define $\varphi_A : \mathbb{Z}/5\mathbb{Z}[x] \longrightarrow M_3(\mathbb{Z}/5\mathbb{Z})$ by

$$p(x) \longmapsto p(A)$$

- $\mathrm{im}(\varphi_A)$ is subring of $M_3(\mathbb{Z}/5\mathbb{Z})$, gen by A.

<u>Claim</u>: $\mathrm{im}(\varphi_A)$ is NOT a field.

Cayley-Hamilton theorem $\Rightarrow$ $c_A(A) = A^3 + A^2 + 3A + 3I_3 = 0$.

$c_A(x) = \text{min poly}$.

check: $A^3 + A^2 + 3A + 3I_3$

$$= \begin{bmatrix} 0 & 4 & 1 \\ 3 & 3 & 0 \\ 4 & 4 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 3 \\ 2 & 1 & 4 \\ 3 & 1 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 1 \\ 0 & 3 & 1 \\ 3 & 0 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

all mod 5

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\leftarrow$ good check to make sure min poly was correct!

So: $\varphi_A(c_A(x)) = \varphi_A(x^3 + x^2 + 3x + 3) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

But $x^3 + x^2 + 3x + 3 = (x+1)(x^2+3)$

So $\varphi_A(c_A(x)) = \varphi_A(x+1)\,\varphi_A(x^2+3) = (A+I)(A^2+3I)$

$A + I_3 = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 2 & 2 \\ 1 & 0 & 2 \end{bmatrix} \neq 0$, $A^2 + 3I_3 = \begin{bmatrix} 4 & 3 & 3 \\ 2 & 4 & 4 \\ 3 & 1 & 1 \end{bmatrix} \neq 0$

So $\mathrm{im}(\varphi_A)$ no domain, hence not field.

(3a) R = ring, $A, C \subseteq B$ submods.

Suppose A and C are finitely generated.

Then $A = (x_1, \dots, x_k)$ and $B = (y_1, \dots, y_n)$.

Claim $A + C = (x_1, \dots, x_k, y_1, \dots, y_n)$.

As $x_i, y_i \in A + C$, we have $\supseteq$.

Let $a + c \in A + C$. Then $a = \sum r_i x_i$, $b = \sum s_i y_i$,

hence $a + c = \sum_{i=1}^{k} r_i x_i + \sum_{i=1}^{n} s_i y_i \in (x_1, \dots, x_k, y_1, \dots, y_n)$,

so $\subseteq$.

(3b) Suppose A, C, and $B/(A+C)$ are finitely generated.

Claim: B is finitely generated.

We have (by (a)) $A + C$ is a f.g. submodule of B with quotient $B/A+C$ f.g., hence B is f.g. (see proof in Thm 19.2.6)

see next page.

~~(3c) Claim, B artinian $\Leftrightarrow$ A, C, $B/(A+C)$ all artinian.~~

~~pf Let $M_1 \supseteq M_2 \supseteq \dots$ be desc chain in B (submods of B).~~

~~Any descending chain in A or C is contained in B, hence stabilizes, thus A and C are artinian.~~

~~Let $M_1 \supseteq M_2 \supseteq \dots$ be a desc. chain in $B/(A+C)$.~~

~~Submods of $B/(A+C)$ have form $N/(A+C)$, N submod of B.~~

~~This gives desc. chain $N_1 \supseteq N_2 \supseteq \dots$ in B ($N_i/(A+C) = M_i$)~~

3c We first prove a claim:

$\boxed{\text{Claim}}$: If $N \subseteq M$ submod and N and M/N artinian, then M is artinian.

Pf of Claim:

Let $L_1 \supseteq L_2 \supseteq \cdots$ be desc. chain in M.

Then $L_1 \cap N \supseteq L_2 \cap N \supseteq \cdots$ desc. chain in N,

$\frac{L_1+N}{N} \supseteq \frac{L_2+N}{N} \supseteq \cdots$ desc. chain in M/N.

As N, M/N artinian, must have $\exists\, n$ such that

both $L_n \cap N = L_{n+1} \cap N = \cdots$

and $\frac{L_n+N}{N} = \frac{L_{n+1}+N}{N} = \cdots$

Let $x \in L_n$. Then $x + N \in \frac{L_n+N}{N} = \frac{L_{n+1}+N}{N}$,

so $\exists \underset{y+n}{\overset{L_{n+1}}{y \in}} \in L_{n+1}+N$ s.t. $x+N = y+n+N = y+N$

Thus $x-y \in N$, so $x-y \in N \cap L_n = N \cap L_{n+1}$

So $x = \underset{\substack{N \cap L_n \\ \| \\ N \cap L_{n+1}}}{(x-y)} + \underset{L_{n+1}}{y}$

$\Rightarrow x \in L_{n+1}$

$\Rightarrow L_n = L_{n+1}$

True $\forall\, i \geq n \Rightarrow L_n = L_{n+1} = \cdots$

$\Rightarrow M$ is artinian.

(3c) ~~[illegible]~~

GOAL: Show B art $\iff$ $A, C, B/(A+C)$ all artinian

$\boxed{\Rightarrow}$ • descending chains in A, C are in B, hence stabilize $\Rightarrow A, C$ artinian.

• ~~[illegible]~~ A descending chain in $B/(A+C)$ lifts to one in B, which stabilizes, hence so does the one in $B/(A+C)$. //

$\boxed{\Leftarrow}$ ~~[illegible]~~ claim

• First: A, C art $\Rightarrow$ $A+C$ art.

$A \subseteq A+C$ and $\frac{A+C}{A} \cong C$, use

so by $\boxed{\text{claim}}$ above, $A+C$ artinian.

• Second, $\underbrace{A+C}_{\text{art.}} \subseteq B$ and $B/(A+C)$ art.

$\boxed{\text{Claim}} \Longrightarrow$ B artinian. ∎