2012 exam solutions

① Find SNF over $\mathbb{Z}$

$$\begin{bmatrix} 2 & 4 & 2 \\ 2 & 4 & 6 \\ 3 & 5 & 7 \end{bmatrix} \xrightarrow[R_3-R_1]{R_2-R_1} \begin{bmatrix} 2 & 4 & 2 \\ 0 & 0 & 4 \\ 1 & 1 & 5 \end{bmatrix} \xrightarrow{R_1-2R_3} \begin{bmatrix} 0 & 2 & -8 \\ 0 & 0 & 4 \\ 1 & 1 & 5 \end{bmatrix}$$

$$\xrightarrow{R_3 \leftrightarrow R_1} \begin{bmatrix} 1 & 1 & 5 \\ 0 & 0 & 4 \\ 0 & 2 & -8 \end{bmatrix} \xrightarrow[C_3-5C_1]{C_2-C_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 2 & -8 \end{bmatrix} \xrightarrow{R_3+2R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 2 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2 \Leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

② $R = \left\{ \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \middle| \, a, b, c \in \mathbb{R} \right\}$

$I = \left\{ \begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix} \middle| \, b \in \mathbb{R} \right\}.$

a) I is ideal: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in I \Rightarrow I \neq \emptyset.$

$$\begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix} \begin{bmatrix} x & 0 \\ y & z \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ bx & 0 \end{bmatrix} \in I$$

$$\begin{bmatrix} x & 0 \\ y & z \end{bmatrix} \begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ zb & 0 \end{bmatrix} \in I$$

$$\begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ b' & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ b+b' & 0 \end{bmatrix} \in I$$

2a continued.

commutative? NO:

$$\begin{bmatrix}1&0\\1&1\end{bmatrix}\begin{bmatrix}2&0\\0&0\end{bmatrix}=\begin{bmatrix}2&0\\2&0\end{bmatrix} \not=$$
$$\begin{bmatrix}2&0\\0&0\end{bmatrix}\begin{bmatrix}1&0\\1&1\end{bmatrix}=\begin{bmatrix}2&0\\0&0\end{bmatrix}$$

semisimple? NO:
$$\begin{bmatrix}0&0\\1&0\end{bmatrix}=\begin{bmatrix}0&0\\1&0\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix} \to R \text{ has nilpotent ideal } (I).$$

artinian / noetherian? Yes:

ideals of R are: I, R, $\left\{\begin{bmatrix}a&0\\b&0\end{bmatrix}\right\}$, $\left\{\begin{bmatrix}0&0\\b&c\end{bmatrix}\right\}$.

If J an ideal, not zero, must have one of $\begin{bmatrix}0&\\x&0\end{bmatrix}$, $\begin{bmatrix}x&0\\0&0\end{bmatrix}$, or $\begin{bmatrix}0&0\\0&x\end{bmatrix}$, ie contains one of $\underset{=I}{\begin{bmatrix}0&0\\\mathbb{R}&0\end{bmatrix}}$, $\begin{bmatrix}\mathbb{R}&0\\0&0\end{bmatrix}$, or $\begin{bmatrix}0&0\\0&\mathbb{R}\end{bmatrix}$.

Note

$$\begin{bmatrix}0&0\\1&0\end{bmatrix}\begin{bmatrix}x&0\\0&0\end{bmatrix}=\begin{bmatrix}0&0\\x&0\end{bmatrix}\in J$$

$$\begin{bmatrix}0&0\\0&x\end{bmatrix}\begin{bmatrix}0&0\\1&0\end{bmatrix}=\begin{bmatrix}0&0\\x&0\end{bmatrix}\in J$$

so J contains either $\begin{bmatrix}0&0\\\mathbb{R}&0\end{bmatrix}$, $\begin{bmatrix}\mathbb{R}&0\\\mathbb{R}&0\end{bmatrix}$, or $\begin{bmatrix}0&0\\\mathbb{R}&\mathbb{R}\end{bmatrix}$.

finite # ideals $\Rightarrow$ noeth + art.

2a cont.

$$R/I \cong \left\{ \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix} \mid a,c \in \mathbb{R} \right\} \cong \mathbb{R} \times \mathbb{R}$$

is commutative, semisimple (by Wedderburn-Artin) hence also noetherian and commutative.

2b $m_1 = \begin{bmatrix} \mathbb{R} & 0 \\ \mathbb{R} & 0 \end{bmatrix}$, $m_2 = \begin{bmatrix} 0 & 0 \\ \mathbb{R} & \mathbb{R} \end{bmatrix}$.

$m_1 \cap m_2 = I$.

2c see 2a.

2d I is simple left R-mod, $\mathbb{R}$ is simple left R-mod.

$$R \to R/I \cong \mathbb{R} \times \mathbb{R} \to \mathbb{R}.$$

$$\begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \cdot r = a \cdot r, \text{ e.g.}$$

~~$\mathbb{R}$ has [illegible]~~

I has nilpotent elmts, $\mathbb{R}$ does not (it is a field), hence $I \ncong \mathbb{R}$.

③

(a) Let R = ring, $I \leq R$ ideal.

Claim: Left R/I-module = left Rmods M such that $IM = 0$.

Pf: • Let M be an R/I-mod.

Let $x \in I$ and $m \in M$.

Then $\underset{\overset{\parallel}{0 \in M}}{(0+I)} \cdot m = (x+I) \cdot m = xm$

$\Rightarrow xm = 0$

$\Rightarrow IM = 0$.

• Let M be R-mod such that $IM = 0$.

Need to show for $r+I = s+I$, $m \in M$ that $rm = sm$.

But $r - s \in I \Rightarrow (r-s)m = 0$

$\Rightarrow rm - sm = 0$

$\Rightarrow rm = sm$ //

(3b) $R = F[x]$, $F =$ field, $I = (x^2)$.

Let M be R/I-module.

Show the following are equivalent:

(i) M f.g. R-mod

(ii) M f.g. R/I-mod.

(iii) M finite dimensional F-vector space (F-mod).

Pf: By (a), $(x^2)M = 0$.

~~(i) $\Rightarrow$ (ii): Let $M = (x_1, \dots, x_n)$ over R~~

~~$= \{r_1 x_1 + \dots r_n x_n \mid r_i \in R\}$.~~

~~(iii) $\Rightarrow$ (ii): Let $x_1, \dots, x_n$ be basis for M over F. Then~~

(iii) $\Rightarrow$ (ii) $\Rightarrow$ (i):

Let $x_1, \dots, x_n$ be F-basis for M.

Then $M = Fx_1 \oplus \dots \oplus Fx_n$.

But then $M = R/I\, x_1 + \dots + R/I\, x_n$

$= R x_1 + \dots + R x_n$,

so M f.g. as R/I-mod and hence as R-mod.

(i) $\Rightarrow$ (iii): Say $M = Rx_1 + \dots + Rx_n$. Then, as $x^2 M = 0$, we have

~~...~~: $M = Fx_1 + \dots + Fx_n + Fx x_1 + \dots + Fx x_n$,

so M is fin. dim over F. //

3c

Let M be a f.g. $F[x]/(x^2)$-mod.

As $F[x]/(x^2) = R/I$ is PID, the decomposition thm $\Rightarrow$

$$M \cong (R/I)^s \oplus \frac{R/I}{R/I(a+bx)} \qquad s \geq 0$$

$$\cong \left(\frac{F[x]}{(x^2)}\right)^s \oplus \frac{F[x]}{(x^2, a_0 + a_1 x)}$$

$a_1 \neq 0$
unit, a_0

$$\cong \left(\frac{F[x]}{(x^2)}\right)^s \oplus F$$

& $s \geq 0$.

$$\frac{F[x]}{(x^2, a_0 + a_1 x)} \cong F$$

R = ring, M left R-module.

4a

Suppose M is noetherian. ~~then~~

Let N be ~~a~~ submod of M.

Suppose N not f.g.

Then $\exists$ $\{x_1, x_2, \ldots\}$ in N

s.t. $(x_1) \subsetneq (x_2) \subsetneq \cdots \subsetneq N$,

but this contradicts ACC, hence N is f.g.

4b Suppose M is noetherian <u>and</u> artinian.

$\exists$ submod $M_1 \subseteq M$ s.t. no submods between M_1, M.

(as noetherian), hence by correspondence thm,

M/M_1 is simple.

Repeat to get desc. chain $M \supseteq M_1 \supseteq M_2 \supseteq \cdots$

where each M_i/M_{i+1} is simple.

This must terminate (at 0) by ~~the~~ DCC.

<u>Ex:</u> $\mathbb{Z}$ is noeth, not artinian,

so $(2) \supseteq (2^2) \supseteq (2^3) \supseteq \cdots$ desc. chain

$$\begin{array}{ccc} (2) & (2^2) & (2^3) \\ \| & \| & \| \\ 2\mathbb{Z}. & 4\mathbb{Z} & 8\mathbb{Z} \end{array}$$

$2^n\mathbb{Z}/2^{n+1}\mathbb{Z}$ simple.