Norwegian University of Science and Technology
Department of Mathematical Sciences

Scientific contakt during the exam:
Aslak Bakke Buan 73 55 02 89/40 84 04 68

Exam in MA3201: Rings and modules

English
December 15. 2012
Tid: 0900-1300

Permitted aids:
simple calculator

All answers should be justified and properly explained.

Problem 1

Find Smith normal form over the integers $\mathbb{Z}$ for the matrix $\begin{bmatrix} 2 & 4 & 2 \\ 2 & 4 & 6 \\ 3 & 5 & 7 \end{bmatrix}$.

Problem 2

Consider the ring $R = \left\{ \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\}$, where $\mathbb{R}$ denotes the real numbers, and the subset $I = \left\{ \begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix} \mid b \in \mathbb{R} \right\}$.

a) Show that I is an ideal in R.

Is the ring R commutative, artinian, noetherian, semisimple? Is the ring R/I commutative, artinian, noetherian, semisimple?

b) Find two maximal ideals m_1, m_2 in R, such that the intersection $m_1 \cap m_2 = I$.

c) Show that there are no other maximal ideals in R.

d) Find two simple R-modules which are not isomorphic.

Problem 3

a) For any ring R and any ideal I in R, show that the left modules over R/I are exactly the left R-modules M such that $IM = 0$.

b) Let $R = F[x]$ for a field F and let $I = (x^2)$. Show that for an R/I-module M, the following three statements are equivalent:

- M is finitely generated as an R-module.
- M is finitely generated as an R/I-module.
- M is finite dimensional as an F-vector space (=F-module).

c) Classify all finitely generated modules over $F[x]/(x^2)$ (up to isomorphism).

Problem 4

Let R be a ring, and let M be a left R-module.

a) Show that if M is noetherian, then all submodules of M are finitely generated.

b) Show that if M is both noetherian and artinian, then there is a finite sequence of submodules

$$M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_{n-1} \supseteq M_n = 0$$

such that M_i/M_{i+1} is a simple R-module, for $i = 0, \ldots, n-1$.

Give an example to show that such a finite sequence does not necessarily exist if M is only noetherian (and not artinian).