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Scientific contact during the exam:
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MA3201 Rings and modules

Thursday 1st December 2011
Time: 09:00–13:00
Permitted aids: Simple calculator

Problem 1 Let $A = \begin{pmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$ in the full matrix ring $M_4(\mathbb{Q})$, where $\mathbb{Q}$ denotes the rational numbers.

a) Find the Smith normal form of the matrix

$$A - xI_4 = \begin{pmatrix} 2-x & 0 & -1 & 0 \\ 0 & 2-x & 0 & -1 \\ 0 & 0 & 2-x & 0 \\ 0 & 0 & 0 & 2-x \end{pmatrix}$$

over $\mathbb{Q}[x]$, where I_4 denotes identity matrix in $M_4(\mathbb{Q})$.

b) Compute the rational canonical form of A.

c) Compute the Jordan canonical form of A.

Problem 2 Let F be a field, and $R = \begin{pmatrix} F & F & F \\ 0 & F & 0 \\ 0 & 0 & F \end{pmatrix}$.

a) Show that R is a subring of the full matrix ring $M_3(F)$.

b) Show that both $I_1 = \begin{pmatrix} 0 & F & 0 \\ 0 & F & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $I_2 = \begin{pmatrix} 0 & F & F \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ are (two-sided) ideals of R.

c) We define a ring to be *semisimple* if it is a finite direct sum of matrix rings over division rings. Give 3 equivalent conditions for a ring R to be semisimple.

d) Is R/I_1 semisimple? Is R/I_2 semisimple? Why or why not?

Problem 3 Let R be a ring (with unity 1). Show that every proper left ideal I is contained in a maximal left ideal of R.

Problem 4 Define a mapping $\iota : \mathbb{C}[t] \to \mathbb{C}[x,y]$ by $\iota(f(t)) = f(x+y) \in C[x,y]$, for $f(t) \in \mathbb{C}[t]$.

a) Show that ι is a homomorphism of rings.

b) Show that if $\psi : R \to S$ is a homomorphism of rings, and M is a left S-module, then M is also a left R-module via the action $r \cdot x = \psi(r)x$, for $r \in R$ and $x \in M$.

c) State the Decomposition Theorem for finitely generated modules over a PID.

d) Consider the ring homomorphism

$$\varphi : \mathbb{C}[t] \to \mathbb{C}[x,y]/(xy)$$

defined as the composition $\varphi = \pi \circ \iota$, where $\pi : \mathbb{C}[x,y] \to \mathbb{C}[x,y]/(xy)$ is the natural projection (φ is also injective, but you do not need to show this). According to Part (b), $\mathbb{C}[x,y]/(xy)$ is itself a module over the ring $\mathbb{C}[t]$ via φ. Show that it is finitely generated as a $\mathbb{C}[t]$-module, and write its decomposition, up to isomorphism, according to the Decomposition Theorem for finitely generated modules over a PID.