Norwegian University of Science and Technology
Department of Mathematical Sciences

Contact during the exam:
Idun Reiten (99 24 45 39)

EXAM IN MA3201 RINGS AND MODULES

Friday December 11, 2009
Time: 09:00 – 13:00
Grades to be announced: Wednesday December 30, 2009
Permitted aids: None.
English

You should give a reason for all answers.

Problem 1

Let F be a field,R $= \left\{ \begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & 0 & e \end{pmatrix} ; a, b, c, d, e \in F \right\}$ and I $= \left\{ \begin{pmatrix} 0 & 0 & 0 \\ b & 0 & 0 \\ d & 0 & 0 \end{pmatrix} ; b,\ d \in F \right\}$

a)

Show that R is a subring of the ring $\mathrm{M}_3(F) = \begin{pmatrix} F & F & F \\ F & F & F \\ F & F & F \end{pmatrix}$ and that I is an ideal in R.

b) Show that the factor ring R/I is isomorphic to the ring $F \times F \times F$. Is R/I a semisimple ring?

c) Is R a left noetherian ring?

d)

$$\text{Let } I_1 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ d & 0 & 0 \end{pmatrix} ; d \in F \right\} \text{ and } I_2 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e \end{pmatrix} ; e \in F \right\}$$

Show that I_1 is a minimal left ideal, and that I_1 and I_2 (which is also a left ideal in R) are isomorphic as R-modules. Find a left ideal I_3 different from I_1 and I_2 such that I_1 and I_3 are isomorphic R-modules.

Problem 2

a) What is meant by the Smith normal form of an $m \times n$-matrix A over a principal ideal domain R? Find the Smith normal form of

$$A = \begin{pmatrix} 6 & 4 & 2 \\ 4 & 2 & 2 \\ 8 & 6 & 6 \end{pmatrix} \text{ over } \mathbb{Z}$$

b) Let $m(x) = (x-2)(x-1)^3$ be the minimal polynomial for a 6×6 - matrix A over $\mathbb{R}$. Find the possibilities for the invariant factors for $A - xI$ over $\mathbb{R}$, and write down the rational canonical form for A in one of the cases.

Problem 3

Let R be a semisimple ring and I an ideal in R. Show that the factor ring R/I is a semisimple ring.

Problem 4

Let A be a left ideal in a ring R, and assume that $A = Aa$ for some $a \neq 0$ in A.

(i) Show that there is some $e \in A$ where $ea \neq 0$ and $(e^2 - e)a = 0$.

(ii) Let $B = \{x \in A; xa = 0\}$. Show that B is a left ideal in R.

(iii) Assume further that the left ideal A is a minimal left ideal. Show that the element e from (i) is then an idempotent element.

(You can use (i) to show (ii) even if you do not show (i), and you can use (i) and (ii) to show (iii)).