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EXAM IN MA3201 RINGS AND MODULES

Thursday Desember 4, 2008
Time: 09.00 – 13:00
Sensurdato: Monday, 5. January 2009
Permitted aids: None.
English

You must give arguments for all your answers.

$\mathbb{R}$ denotes the real numbers and $\mathbb{Z}$ denotes the integers.

Problem 1

Let F be a field and

$$R = \begin{pmatrix} F & 0 & 0 \\ F & F & 0 \\ F & 0 & F \end{pmatrix}$$

a) Show that R is a ring and that $I = \begin{pmatrix} 0 & 0 & 0 \\ F & 0 & 0 \\ F & 0 & 0 \end{pmatrix}$ is an ideal in R.

b) Show that the factor ring R/I is isomorphic to the ring $F \times F \times F$. Is R/I a semisimple ring?

c) Show that $I_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ F & 0 & 0 \end{pmatrix}$ and $I_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & F & 0 \\ 0 & 0 & 0 \end{pmatrix}$

are minimal left ideals in R, and that I_1 and I_2 are not isomorphic R-modules.

Problem 2

a) Let R be a ring and M a noetherian R-module. Let N be a submodule of M. Show that the factor module M/N is a noetherian R-module.

b) Show that the ring of integers $\mathbb{Z}$ is not an artinian ring.

Problem 3

Let R be a ring and I a left ideal in R. Assume there is a left ideal J in R such that $R = I \bigoplus J$. Show that $I = Re$, where e is an idempotent.

Let F be a field and $I = \begin{pmatrix} F & 0 \\ F & 0 \end{pmatrix}$ a left ideal in the ring $\begin{pmatrix} F & 0 \\ F & F \end{pmatrix}$. Find an idempotent e in R such that $I = Re$, and a left ideal J in R such that $R = I \bigoplus J$.

Problem 4

Find a nonzero nilpotent ideal in the ring $\mathbb{Z}/(4)$. For which $n \geq 1$ is the ring

$$\begin{pmatrix} Z/(2^n) & Z/(2^n) \\ Z/(2^n) & Z/(2^n) \end{pmatrix} \text{ semisimple?}$$

Problem 5

a) Find Smith normal form over $\mathbb{Z}$ of the matrix

$$A = \begin{pmatrix} 0 & 2 & -1 \\ -3 & 8 & 3 \\ 2 & -4 & -1 \end{pmatrix}$$

b) Let A be a 6×6 matrix over $\mathbb{R}$ with minimal polynomial $m(x) = (x^2+1)(x-2)(x-1)$. Find all possibilities for the non-unit monic invariant factors for the matrix $A - xI_6$. In each case, find the corresponding rational canonical form for A.

c) Let V be a vector space over $\mathbb{R}$ of dimension 4, and let $T : V \to V$ be a linear transformation. View V(with T) as $\mathbb{R}[x]$-module in the usual way. Assume that $\{v_1, v_2, Tv_2, T^2v_2\}$ is a basis for the vector space V, for some v_1, v_2 in V, and that $Tv_1 = v_1$ and $T^3v_2 = 4T^2(v_2) - 5T(v_2) + 2v_2$. Find $f_1(x)$ and $f_2(x)$ in $\mathbb{R}[x]$, with $f_1(x)|f_2(x)$ such that $V \simeq \mathbb{R}[x]/(f_1(x)) \bigoplus \mathbb{R}[x]/(f_2(x))$ as $\mathbb{R}[x]$-modules.