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Faglig kontakt under eksamen:
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EXAM IN RINGS AND MODULES (MA3201)

Tuesday, 11.th December 2007
Time: 09:00 – 13:00
Grades to be announced: Friday, 21 December 2007
Permitted aids: None.

You should give a reason for all answers.

Problem 1

Let F be a field , R the matrix ring

$$\mathrm{R} = \begin{pmatrix} F & 0 & 0 \\ F & F & 0 \\ F & F & F \end{pmatrix}$$

and

$$\mathrm{I} = \begin{pmatrix} 0 & 0 & 0 \\ F & 0 & 0 \\ F & F & 0 \end{pmatrix}$$

a) Show that I is an ideal in R, and that I is nilpotent. Is R a semisimple ring?

b) Show that the factor ring R/I is a semisimple ring.

c) Find 2 different minimal left ideals in R which are isomorphic as R-modules.

d) Find all the ideals in R which contain the ideal I.

Problem 2

Let $\mathbb{Z}$ be the ring of integers, and let R be the ring $\begin{pmatrix} \mathbb{Z} & 0 \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix}$

a) Find all idempotent elements in R, and describe the left ideals of the form Re for an idempotent element e in R.

b) Let I be the left ideal $\begin{pmatrix} \mathbb{Z} & 0 \\ \mathbb{Z} & 0 \end{pmatrix}$. Find an infinite number of left ideals J in R such that $R = I \oplus J$.

Problem 3

a) Show that the ring of integers $\mathbb{Z}$ is noetherian, and not artinian.

b) Give a proof of the fact that if M is a noetherian module over a ring R, then M is a finitely generated R-module.

Problem 4

a) Denote by $\mathbb{R}$ the real numbers. Find the Smith normal form over $\mathbb{R}[x]$ for the matrix

$$\begin{pmatrix} -3-x & 2 & 0 \\ 1 & -x & 1 \\ 1 & -3 & -2-x \end{pmatrix}$$

Let $V = \mathbb{R}^3$, and let $T = T_A : V \to V$ be the linear transformation given by the matrix $A = \begin{pmatrix} -3 & 2 & 0 \\ 1 & 0 & 1 \\ 1 & -3 & -2 \end{pmatrix}$ with respect to the standard basis for $V = \mathbb{R}^3$. Describe the $\mathbb{R}[x]$-module V (defined using $T : V \to V$) in terms of cyclic $\mathbb{R}[x]$-modules.

b) Let A be a 7 x 7 matrix over $\mathbb{R}$, with characteristic polynomial $c(x) = -(x-1)^2(x-2)^3(x^2+1)$ and with minimal polynomial $m(x)$ of degree 5. Find all the possibilities for the invariant factors for A, (that is, for $xI - A$), and in each case, the associated rational canonical form for A.