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EXAM IN RINGS AND MODULES (MA3201)

English
Friday 15th December 2006
Time: 09:00–13:00
Permitted aids: None

Grades: 15.01.2007.

Problem 1 Let A be the 3×3 matrix

$$\begin{pmatrix} 1 & 2 & -4 \\ 1 & 2 & 2 \\ -1 & 1 & 1 \end{pmatrix}$$

over $\mathbb{C}$, the complex numbers.

a) Find the Smith normal form of the matrix $A - xI_3$ over the ring $\mathbb{C}[x]$, where $\mathbb{C}[x]$ is the polynomial ring in one variable x over $\mathbb{C}$ and I_3 is the 3×3 identity matrix.

b) Find the rational canonical form of the matrix A over $\mathbb{C}$.

c) Find the Jordan canonical form of the matrix A over $\mathbb{C}$.

Problem 2 Let R and S be two rings. An abelian group M is called a S-R-bimodule if M is a left S-module and a right R-module, such that

$$s(mr) = (sm)r$$

for all s in S, for all r in R and for all m in M. Let

$$\Lambda = \begin{pmatrix} R & 0 \\ M & S \end{pmatrix}$$

where M is a S-R-bimodule different from (0). Let $\begin{pmatrix} r & 0 \\ m & s \end{pmatrix}$ and $\begin{pmatrix} r' & 0 \\ m' & s' \end{pmatrix}$ be two elements in Λ. The set Λ becomes an abelian group under the binary operation, $+$, given by

$$\begin{pmatrix} r & 0 \\ m & s \end{pmatrix} + \begin{pmatrix} r' & 0 \\ m' & s' \end{pmatrix} = \begin{pmatrix} r+r' & 0 \\ m+m' & s+s' \end{pmatrix}.$$

Define a binary operation, $\cdot$, on Λ by letting

$$\begin{pmatrix} r & 0 \\ m & s \end{pmatrix} \cdot \begin{pmatrix} r' & 0 \\ m' & s' \end{pmatrix} = \begin{pmatrix} rr' & 0 \\ mr'+sm' & ss' \end{pmatrix}.$$

a) Show that Λ is a ring with 1, when addition, $+$, and multiplication, $\cdot$, is defined as above.

b) Find

(i) an idempotent element different from 0 and 1 in Λ,

(ii) a nilpotent element different from 0 i Λ.

c) Let $I = \{\begin{pmatrix} 0 & 0 \\ m & 0 \end{pmatrix} \mid m \in M\}$. Show that I is a twosided ideal in Λ. Show that $\Lambda/I \simeq R \oplus S$ as rings.

Problem 3 Let k be a field. The map $\varphi\colon k[x]/(x^2) \to k$ given by

$$\varphi(f(x) + (x^2)) = f(0)$$

is a homomorphism of rings. Let $R = k$ and $S = k[x]/(x^2)$.

a) Let M be a left R-module. Show that M becomes a left S-module by defining

$$s \cdot m = \varphi(s)m$$

for all s in S and for all m in M.

b) Let $M = k^2 = \{(a, b) \mid a, b \in k\}$. Then is M a left k-module by letting

$$\alpha(a, b) = (\alpha a, \alpha b)$$

and a right k-module by letting

$$(a, b)\alpha = (a\alpha, b\alpha)$$

for all α in k and for all (a, b) in M. With these module structures M becomes a k-k-bimodule (Do not need to show this). By a) we have that the left k-module M is a left S-module by letting $(f(x) + (x^2)) \cdot m = \varphi(f(x) + (x^2))m$. Show that M is a S-R-bimodule, when $R = k$ and $S = k[x]/(x^2)$.

c) Now let $\Lambda = \left(\begin{smallmatrix} R & 0 \\ M & S \end{smallmatrix}\right)$, where M is as in b), and Λ is a ring as given in Problem 2. Show that Λ is an algebra over k. What is $\dim_k \Lambda$? Decide if Λ is

(i) a left artinian ring,

(ii) a left noetherian ring,

(iii) a semisimple ring.

d) Let J be the left ideal $\{\left(\begin{smallmatrix} 0 & 0 \\ (0,a) & bx+(x^2) \end{smallmatrix}\right) \mid a, b \in k\}$. Consider the left Λ-module $X = \Lambda/J$. Show that $f\colon X \to X$ given by

$$f(\lambda + J) = \lambda \left(\begin{smallmatrix} 0 & 0 \\ (0,0) & 1+(x^2) \end{smallmatrix}\right) + J$$

is a Λ-homomorphism. Find the image $\operatorname{Im} f$ of f. Show that $X = \operatorname{Im} f \oplus Y$ for a submodule Y of X.