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MNFMA318, Rings and modules

English
Saturday, Desember 7, 2002
Time: 9-13
Permitted aids: None
Grades to be announced: Monday, January 6, 2003

Problem 1
Let $\mathbb{R}$ denote the field of real numbers and let

$$R = \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ b & c & 0 & 0 \\ d & 0 & e & 0 \\ f & 0 & 0 & g \end{pmatrix} ; a, b, c, d, e, f, g \in \mathbb{R} \right\} \quad \text{and} \quad I = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ b & 0 & 0 & 0 \\ d & 0 & 0 & 0 \\ f & 0 & 0 & 0 \end{pmatrix} ; b, d, f \in \mathbb{R} \right\}$$

a) Show that R is a ring and that I is an ideal in R.

b) Show that R/I and $\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ are isomorphic rings.

c) Is R a semisimple ring? Is R/I a semisimple ring?

d) Find 3 minimal left ideals in R.

Problem 2
Let R be a ring and M an R-module. Let N and L be submodules of M.

a) Show that $N \cap L$ is a submodule of M, and give an example to show that $N \cup L$ is not always a submodule of M.

b) Assume that M is a noetherian R-module. Show that then N and M/N are noetherian R-modules.

Problem 3

a) Find the possible invariant factors and rational canonical forms for 6×6-matrices over the real numbers $\mathbb{R}$, with minimal polynomial $(x^2 + 1)(x - 3)^2$.

b) Let V be a vector space of dimension 4 over the real numbers $\mathbb{R}$, and let $T : V \to V$ be a linear transformation. Let v_1, v_2 be elements in V such that $\{v_1, v_2, Tv_2, T^2 v_2\}$ is a basis for V, and assume that $Tv_1 = 2v_1$ and $T^3 v_2 = 2v_2 + 3Tv_2$. In the usual way we view V (together with $T : V \to V$) as an $\mathbb{R}[x]$-module. Show that v_1 and v_2 generate V as an $\mathbb{R}[x]$-module, and find $f_1(x)$ and $f_2(x)$ in $\mathbb{R}[x]$, where $f_1(x) | f_2(x)$, such that $V \simeq \mathbb{R}[x]/(f_1(x)) \oplus \mathbb{R}[x]/(f_2(x))$.