Løsningsforslag Ma 3201 24/11-20

1a

$$\left|\begin{pmatrix} x-7 & -2 & 12 \\ -4 & x-5 & 12 \\ -2 & 1 & x+3 \end{pmatrix}\right| = (x-7)(x-5)(x+3) + 48 - 48$$

$$+ 24(x-5) - 12(x-7) - 8(x+3)$$

$$= x^3 - 9x^2 + 3x + 45 = (x-3)(x^2-6x-15)$$

$$= (x-3)\left((x-3)^2 - 24\right) = (x-3)(x-3-2\sqrt{6})(x-3+2\sqrt{6})$$

Therefor the Smith normal form of this matrix is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & x^3-9x^2+3x+45 \end{pmatrix}$$

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b) ~~Fi~~

$$\left|\begin{pmatrix} x-7 & -2 & 12 \\ -4 & x-5 & 12 \\ -2 & -1 & x+3 \end{pmatrix}\right| = (x-3)^3$$

The rank of the matrix $\begin{pmatrix} 3-7 & -2 & 12 \\ -4 & 3-5 & 12 \\ -2 & -1 & 6 \end{pmatrix}$

is 1, Hence the Jordan Canonical form is

$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 1 & 3 \end{pmatrix}$ and the rational canonical form is

$$\left(\begin{array}{c|cc} 3 & 0 & 0 \\ \hline 0 & 0 & -9 \\ 0 & 1 & 6 \end{array}\right).$$

Problem 2

$$\Lambda_1 = \left\{ a\underbrace{\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}}_{I} + b\underbrace{\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}}_{B} + c\underbrace{\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}}_{C} + d\underbrace{\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}}_{D} \right.$$

$$\left. a,b,c,d \in \mathbb{Q} \right\}$$

Let the three matrices be called I, B, C, D respectively. Then $1 = I^2 = B^2 = C^2 = D^2$

$CB = BC = D \quad BD = C = DB \quad CD = DC = B.$

Hence the product of any linear combination give again a linear combination of the four matrices and is hence closed with respect to multiplication. Λ_1 is a 4-dimensional $\mathbb{Q}$-vectorspace so clearly an abelian group under addition. Hence Λ_1 is a subring of the full 4×4 matrix ring over $\mathbb{Q}$.

$$\Lambda_2 = \{ aI + bB + cC + dD \mid a,b,c,d \in \mathbb{Z}_2 \}$$

The same multiplication table for the matrices I, B, C, D is valid, and hence Λ_2 is closed with respect to multiplication in the ring of 4×4-matrices over $\mathbb{Z}_2$. Λ_2 is also a 4-dimensional vectorspace over $\mathbb{Z}_2$, and hence an abelian group under usual addition. So Λ_2 is a subring.

Since $(aI + bB + cC + dD)^2 = (a^2+b^2+c^2+d^2)I$ + a vector ~~of B, C, D~~ is in the vectorsubspace spanned by B, C and D, we have that if $\alpha \in \Lambda, \alpha \neq 0 \Rightarrow \alpha^2 \neq 0$. Hence $\alpha^4 \neq 0$ and so on, so α is not nilpotent

b) $\varphi_1(aI + bB + cC + dD) = a + b + c + d$

$\alpha = aI + bB + cC + dD \quad \alpha' = a'I + b'B + c'C + d'D$

$\varphi_1(\alpha + \alpha') = \varphi((a+a')I + (b+b')B + (c+c')C + (d+d')D)$

$= (a+a') + (b+b') + (c+c') + (d+d') = a + b + c + d + a' + b' + c' + d'$

$= \varphi_1(\alpha) + \varphi_1(\alpha')$

$\varphi_1(\alpha \cdot \alpha') = \varphi_1\big((aa' + bb' + cc' + dd')I + (ab' + ba' + cd' + dc')B$

$+ (ac' + ca' + bd' + db')C + (ad' + da' + bc' + cb')D\big)$

$= (aa' + bb' + cc' + dd') + (ab' + ba' + cd' + dc')$

$+ (ac' + ca' + bd' + db') + (ad' + da' + bc' + cb')$

$= (a+b+c+d)(a'+b'+c'+d') = \varphi_1(\alpha)\,\varphi_1(\alpha')$

So φ_1 is a ring homomorphism

The same calculation is valid for φ_2 so φ_2 is also a ring homomorphism

c) Since both φ_1 and φ_2 are surjective, the kernel of both are of dimension 3 over $\mathbb{Q}$ and $\mathbb{Z}_2$ respectively. A basis for the Kernel of φ_1 is $I - B$, $I - C$, $I - D$ and basis for the kernel for φ_2 is $I + B$, $I + C$, $I + D$.

Now $(a(I+B) + b(I+C) + c(I+D))^2 = a^2(I+B)^2 + b^2(I+C)^2 + c(I+D)^2$

$= 0$, so $\ker\varphi_2$ is nil and therefore nilpotent

In fact $(\ker\varphi_2)^3 = 0$.

d) $\psi_1 (aI + bB + cC + dD) = (a+b-c-d,\ a-b+c-d)$

is a surjective ring homomorph is straight forward

The isomorphism $\varphi : \Lambda_1 \to \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$ is

given by $\varphi (aI + bB + cC + dD) = (a+b+c+d,\ a+b-c-d,$
$a-b+c-d,\ a-b-c+d)$.

Problem 3

a) $A_2 = \{ e^{\frac{l 2\pi i}{2^k}} \mid k, l \in \mathbb{N} \}$.

The elements in A_2 of order 2^k are the elements $e^{\frac{l 2\pi i}{2^k}}$ where $\gcd(l, 2) = 1$.

~~Each of~~ them generate ~~s of~~ the cyclic group $\langle e^{\frac{2\pi i}{2^k}} \rangle$. and this group contains all elements of order a divisor of 2^k.

Now let G be a subgroup of order 2^k. If G is not cyclic, all elements of G is contained in $e^{\frac{2\pi i}{2^{k-1}}}$ which is a contradiction. Hence G is cyclic and therefore contain an element of order 2^k which are then a generator for $\langle e^{\frac{2\pi i}{2^k}} \rangle$, so that is the only group of order 2^k.

b) 1) $\langle 1 \rangle \subsetneq \langle e^{\frac{2\pi i}{2}} \rangle \subsetneq \langle e^{\frac{2\pi i}{4}} \rangle \subseteq \dots$ is an accending sequence of subgroups that never stops, so A_2 is not noethenian.

2) Let G be a proper subgroup of A_2

$\exists\ e^{\frac{l 2\pi i}{2^k}} \notin G$. Choose k minimal with this property. Then G contain no element of order 2^k, and hence all elements of G has order at most 2^{k-1}. Then G is contained in $\langle e^{\frac{2\pi i}{2^{k-1}}} \rangle$ which is a finite group, so G is a finite group.

Now, if $A_2 \supseteq G_1 \supseteq G_2 \supseteq \dots$ is a decending sequence of subgroups, Then are two possibilities All $G = A_2$ and the sequence is stable, or at least one $G_i \neq A_2$ G_i is then finite and the sequence stabilizes since G_i is artinian.

c) $\varphi: A_2 \to A_2$ a group homomorphism

If $\varphi \neq 0$ $\ker\varphi$ is a proper subgroup so finite
Then the image of φ have to be ∞, hence the whole of A_2.

Now $End(A_2)$ $f, g \in End\, A_2$ $f \neq 0 \neq g$ then
$f \circ g$ is surjective since g is surjective and f is surjective, Hence $f \circ g \neq 0$.

4 M artinian and $f: M \to M$ injective

$M \supseteq Im f \supseteq Im f^2 \supseteq \dots \supseteq Im f^n \supseteq Im f^{n+1}$ stabilize
so $Im f^n = Im f^{n+1}$ $a \in M$ $f^n(a) \in Im f^n = Im f^{n+1}$
$\exists$ $b \in M$ with $f^n(a) = f^{n+1}(b)$. But f injective $\Rightarrow$ f^2 injective
f^n injective $f^n(a) = f^n f(b)$ $a = f(b)$ i.e $a \in Im f$ so
$M = Im f$.