

Notes: All rings have unity (all rings have a 1). \mathbb{Z} denotes the ring of integers and \mathbb{R} denotes the field of real numbers. All answers must be justified. Each sub-problem is worth the same for grading.

Problem 1. Let R be a ring.

- (1.a) Prove that the polynomial ring $R[x]$ is an R -module, and show that it is free as an R -module.
- (1.b) If A and B are R -submodules of an R -module M , show that there is an R -module isomorphism: $(A + B)/A \cong B/(A \cap B)$.
- (1.c) Prove that if J is a maximal ideal of R , then J is a prime ideal.

Problem 2. Let F be any field.

- (2.a) Prove that the polynomial ring $F[x]$ is noetherian, but that it is not artinian.
- (2.b) Consider the ring $R = \begin{bmatrix} F & F & F \\ 0 & F & F \\ 0 & 0 & F \end{bmatrix}$ and show that $I = \begin{bmatrix} 0 & F & F \\ 0 & 0 & F \\ 0 & 0 & 0 \end{bmatrix} \subseteq R$ is an ideal. Determine whether R and R/I are semisimple rings (and justify).

Problem 3. Let R be a ring.

- (3.a) The annihilator of an R -module M is $\text{Ann}(M) = \{r \in R \mid rM = 0\}$. If M and N are R -modules, prove that if $M \cong N$, then $\text{Ann}(M) = \text{Ann}(N)$. Give an example to show the converse of this statement is false.
- (3.b) Let R be a principal ideal domain and let M be a finitely generated torsion R -module. Prove that M is irreducible (or simple) if and only if $M = Rx$ for any nonzero element $x \in M$ such that $\text{Ann}(Rx)$ is a prime ideal.

Problem 4.

- (4.a) Determine the Smith normal form of the matrix $A = \begin{bmatrix} 3 & -4 & -6 \\ 3 & 3 & 0 \\ 3 & 2 & 0 \end{bmatrix}$ over \mathbb{Z} .
- (4.b) Let B and C be 2×2 matrices over \mathbb{R} . Prove that B is similar to C if and only if their minimal polynomials are equal, that is, $m_B(x) = m_C(x)$.
- (4.c) Let D be a 6×6 matrix with entries in \mathbb{R} and with characteristic polynomial $c_D(x) = (x - 2)^3(x - 5)^3$ and minimal polynomial $m_D(x) = (x - 2)(x - 5)^2$. Determine the rational canonical form of D , the Jordan canonical form of D , and determine whether D is similar to a diagonal matrix.

MA3201 Exam - Fall 2019 - Possible solutions / sketch

- (1a)
- $R[x]$ is a ring, hence an abelian group under addition.
 - For $r_1, r_2 \in R$, $p(x), q(x) \in R[x]$,
 - $r_1(p(x) + q(x)) = r_1 p(x) + r_1 q(x)$
 - $(r_1 + r_2)p(x) = r_1 p(x) + r_2 p(x)$
 - $(r_1 r_2)p(x) = r_1(r_2 p(x))$
 - $\text{if } p(x) = q(x)$

} from operations
in $R[x]$.

Thus $R[x]$ is an R -module via scalar multiplication
from R .

To show $R[x]$ is a free R -module, note that $\{1, x, x^2, \dots\}$
is both linearly independent and generates (it's a basis) over R ,
so $R[x] \cong \bigoplus_{i=0}^{\infty} Rx^i$.
as R -modules.

- (1b) Define $\varphi: A+B \rightarrow B/(A \cap B)$
- $$a+b \mapsto b + A \cap B$$

- It is onto by definition.
- Check it is well-defined and a R -homomorphism.
- Show $\ker \varphi = A$:

If $a \in A$, then $\varphi(a) = \varphi(a+0) = 0 + A \cap B$.
 $\Rightarrow a \in \ker \varphi$.

If $a+b \in \ker \varphi$, then $a+b \notin A \cap B$
 $\Rightarrow b \in A$.

- Fundamental Isomorphism theorem $\Rightarrow \frac{A+B}{A} \cong \text{im } \varphi = \frac{B}{(A \cap B)}$.

(1c) Let $J = \text{max ideals of } R$. If A, B are ideals such that $AB \subseteq J$, and $A \neq J$, then maximality $\Rightarrow A+J=R$. Thus we may find $a \in A, x \in J$ such that $a+x=1$. For any $b \in B$, $ab + x b = 1 b = b$,
 hence $b \in J \wedge b \in A \Rightarrow b \in J$.
 Thus J is prime.

(2a) F is a field, hence noetherian, so Hilbert's basis theorem $\Rightarrow F[x]$ is noetherian. (Alternative: $F[x]$ is P ID \Rightarrow noeth.)
 $F[x]$ is not artinian, as there exists a properly descending chain of ideals: $(x) \supsetneq (x^2) \supsetneq (x^3) \supsetneq \dots$

(2b) To show I is an ideal, check it is nonempty, closed under subtraction, and for any $A \in R$, both $IA, A \in I \subseteq R$ (show this).
 Also, show $I^3 = 0$, hence R is not semisimple because it has a nonzero nilpotent ideal (by Wedderburn-Artin).

On the other hand, $R/I = \begin{bmatrix} F & 0 & 0 \\ 0 & F & 0 \\ 0 & 0 & F \end{bmatrix} \cong F \times F \times F$

$\Rightarrow R/I$ is semi-simple by W-A.

(3a) Let $\varphi: M \rightarrow N$ be an isomorphism. If $r \in \text{ann} M$, then for $n \in N$, there exists $m \in M$ with $\varphi(m) = n$, hence $r \cdot n = r \varphi(m) = \varphi(rm) = \varphi(0) = 0$.

For converse, note that the \mathbb{Z} -modules $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ are not isomorphic but both have annihilator $= 2\mathbb{Z}$.

- (3b) If M is simple, then for any nonzero $x \in M$, $Rx \leq M$ is a nonzero submodule, hence $Rx = M$.
- Assume $M = Rx$ for any nonzero $x \in M$ such that $\text{Ann}(Rx)$ is prime.
- As M is f.g. torsion over a PID, we have
- $$M \cong R/\left(p_1^{a_1}\right) \times \cdots \times R/\left(p_t^{a_t}\right), \text{ some primes } p_i, \text{ integers } a_i.$$
- Write $M = Rx_1 \oplus \cdots \oplus Rx_t$, where $\text{Ann}(Rx_i) = (p_i^{a_i})$.
- Note that $R(p_1^{a_1-1}x_1) \subseteq M$ with $\text{Ann}(R(p_1^{a_1-1}x)) = p_1$, which is prime.
- Thus $M = R_{p_1^{a_1}}x_1 \Rightarrow M \cong R/(p_1)$.
- As prime ideals are maximal in a PID, we obtain that $R/(p_1)$ has no nontrivial left ideals (as a ring) hence M has no nontrivial (left) R -submodules $\Rightarrow M$ is simple. \square

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$$\begin{bmatrix} 3 & -4 & -6 \\ 3 & 3 & 0 \\ 3 & 2 & 0 \end{bmatrix} \xrightarrow[R_2-R_3]{\downarrow R_2} \begin{bmatrix} 3 & -4 & -6 \\ 0 & 1 & 0 \\ 3 & 2 & 0 \end{bmatrix} \xrightarrow[R_3-R_1]{\downarrow R_3} \begin{bmatrix} 3 & -4 & -6 \\ 0 & 1 & 0 \\ 0 & 6 & 6 \end{bmatrix}$$

$$\xrightarrow[C_1 \leftrightarrow C_2]{} \begin{bmatrix} -4 & 3 & -6 \\ 1 & 0 & 0 \\ 6 & 0 & 6 \end{bmatrix} \xrightarrow[R_1 \leftrightarrow R_2]{} \begin{bmatrix} 1 & 0 & 0 \\ -4 & 3 & -6 \\ 6 & 0 & 6 \end{bmatrix} \xrightarrow[R_2+4R_1 \rightarrow R_2]{R_3-6R_1 \rightarrow R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -6 \\ 0 & 0 & 6 \end{bmatrix}$$

$$\xrightarrow[C_3+2C_2 \rightarrow C_3]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}, \text{ note } 1 \mid 3 \mid 6 \text{ in } \mathbb{Z}.$$

(4b) • Assume first B is similar to C . Then RCF of B and C is the same, so B and C have same invariant factors, hence and in particular, the same minimal polynomial.

• Now assume $m_B(x) = m_C(x)$.

case 1: $m_B(x) = m_C(x)$ is quadratic, so $\alpha = x^2 + \alpha x + \beta$.

\Rightarrow RCF of B and C is $\begin{bmatrix} 0 & -\beta \\ 1 & -\alpha \end{bmatrix}$,

hence B and C are similar.

case 2: $m_B(x) = m_C(x)$ is linear, so $\alpha = x - 8$.

Since $C_A(x) = \text{product of invariant factors}$ is degree 2, and all invariant factors divide $m_A(x)$, must have $C_A(x) = (x-8)^2$.

\Rightarrow RCF of A is $\begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$,

and similarly for RCF of B , hence B is similar to A .

$$(4c) \text{ inv. factors: } x-2, \underbrace{(x-2)(x-5)}_{x^2-7x+10}, \frac{(x-2)(x-5)^2}{= x^3 - 12x^2 + 45x - 50}$$

$$\text{elem. div: } x-2, x-2, x-5, x-2, (x-5)^2.$$

$$\text{RCF: } \left[\begin{array}{r} 2 \\ \hline 0 & -10 \\ \hline 1 & 7 \\ \hline 0 & 0 & 50 \\ \hline 1 & 0 & -45 \\ \hline 0 & 1 & 12 \end{array} \right]$$

$$\text{JCF: } \left[\begin{array}{r} 2 \\ \hline 2 & 2 \\ \hline 5 \\ \hline 2 \\ \hline 5 & 1 & 5 \end{array} \right]$$

Since JCF is not diagonal, the matrix D is not diagonalizable.