

PROBLEMS ABOUT PRINCIPAL IDEAL DOMAINS

Let R be a PID.

- (1) Let T be a torsion R -module and let p be a prime of R . Prove that if $px = 0$ for some nonzero $x \in T$, then $\text{Ann}(T) \subseteq (p)$.
- (2) Give an example of a PID R and a nonzero torsion R -module M such that $\text{Ann}(M) = 0$.
- (3) If M is a finitely generated torsion R -module, prove $\text{Ann}(M) \neq 0$.
- (4) If M is a finitely generated R -module, prove that $M/\text{Tors}(M)$ is a free R -module.
- (5) Consider the \mathbb{Z} -module $M = \mathbb{Z}^3 \oplus (\mathbb{Z}/10\mathbb{Z}) \oplus (\mathbb{Z}/24\mathbb{Z})$. What are the invariant factors of M ? What are the elementary divisors of M ? What is $\text{Tors}(M)$ isomorphic to? What is the free rank of M ?
- (6) If M and N are isomorphic R -modules, show that $\text{Ann}(M) = \text{Ann}(N)$.
- (7) Let $M = F[x]/(a(x))$, for a field F and a monic polynomial given by $a(x) = x^k + b_{k-1}x^{k-1} + \cdots + b_1x + b_0$. For a polynomial $p(x) \in F[x]$, denote by $\overline{p(x)} = p(x) \bmod (a(x))$. Prove that $\overline{1}, \overline{x}, \overline{x^2}, \dots, \overline{x^{k-1}}$ is a basis for the F -vector space M .
- (8) Find the Smith Normal Form of the matrix $A = \begin{bmatrix} 0 & 2 & -1 \\ -3 & 8 & 3 \\ 2 & -4 & -1 \end{bmatrix}$ over \mathbb{Z} .
- (9) Find the Rational Canonical Form of the matrix $B = \begin{bmatrix} -3 & 2 & 0 \\ 1 & 0 & 1 \\ 1 & -3 & -2 \end{bmatrix}$ over \mathbb{Q} .
- (10) Explain why all 2×2 matrices with characteristic polynomial $x^2 - 1$ over \mathbb{C} are similar to each other.
- (11) Find all similarity classes of 3×3 matrices with characteristic polynomial $(x - 2)^3$ over \mathbb{Q} .
- (12) Find all similarity classes of 6×6 matrices over \mathbb{C} with characteristic polynomial $(x^4 - 1)(x^2 - 1)$.
- (13) Find the rational canonical form of the matrix $C = \begin{bmatrix} 0 & -1 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$.
- (14) Find the Jordan canonical form of the matrix $D = \begin{bmatrix} 9 & 4 & 5 \\ -4 & 0 & -3 \\ -6 & -4 & -2 \end{bmatrix}$.
- (15) Find the Jordan canonical form of the matrix $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}$.

- (16) Find all Jordan canonical forms of 2×2 , 3×3 , and 4×4 matrices over \mathbb{C} .
- (17) Determine all possible Jordan canonical forms for a matrix with characteristic polynomial $(x - 2)^3(x - 3)^2$.