Further,

$$\sigma(\phi)\sigma(\psi) = (\pi_i\phi\lambda_j)(\pi_i\psi\lambda_j) = \left(\sum_{l=1}^{k} \pi_i\phi\lambda_l\pi_l\psi\lambda_j\right),$$

by definition of multiplication of matrices. But since $\Sigma_{l=1}^{k}\lambda_l\pi_l = 1$, it follows that

$$\sigma(\phi)\sigma(\psi) = (\pi_i\phi\psi\lambda_j) = \sigma(\phi\psi).$$

Therefore, σ is a homomorphism.

To prove that σ is injective, let $\sigma(\phi) = (\pi_i\phi\lambda_j) = 0$. Then $\pi_i\phi\lambda_j = 0$, $1 \leq i,j \leq k$. This implies $\Sigma_{i=1}^{k}\pi_i\phi\lambda_j = 0$. But since $\Sigma_{i=1}^{k}\pi_i = 1$, we obtain $\phi\lambda_j = 0$, $1 \leq j \leq k$. In a similar fashion we get $\phi = 0$, which proves that σ is injective. To prove that σ is surjective, let $f = (f_{ij}) \in T$, where $f_{ij}: M_j \to M_i$ is an R-homomorphism. Set $\phi = \Sigma_{i,j}\lambda_i f_{ij}\pi_j$. Then $\phi \in \mathrm{Hom}_R(M,M)$. By definition of σ, $\sigma(\phi)$ is the $k \times k$ matrix whose (s,t) entry is $\pi_s(\Sigma_{i,j}\lambda_i f_{ij}\pi_j)\lambda_t = f_{st}$, because $\pi_p\lambda_q = \delta_{pq}$. Hence, $\sigma(\phi) = (f_{st}) = f$. Thus, σ is also surjective. □

Problems

1. Let $M = M_1 \oplus M_2$ be the direct sum of simple modules M_1 and M_2 such that $M_1 \not\simeq M_2$. Show that the ring $\mathrm{End}_R(M)$ is a direct sum of division rings. [*Hint:* $\mathrm{Hom}_R(M_1,M_2) = 0$, etc.]
2. Let $M = M_1 \oplus M_2$ be the direct sum of isomorphic simple modules M_1, M_2. Show that $\mathrm{End}_R(M) \simeq D_2$, the 2×2 matrix ring over a division ring.

2 Noetherian and artinian modules

Recall that an R-module M is finitely generated if M is generated by a finite subset of M; that is, if there exist elements $x_1,\ldots,x_n \in M$ such that $M = (x_1,\ldots,x_n)$. This is equivalent to the statement: If $M = \Sigma_{\alpha\in\Lambda}M_\alpha$ is a sum of submodules M_α then there exists a finite subset Λ' of Λ such that $M = \Sigma_{\alpha\in\Lambda'}M_\alpha$. We now define a concept that is dual to that of a finitely generated module.

Definition. *An R-module M is said to be* finitely cogenerated *if, for each family* $(M_\alpha)_{\alpha\in\Lambda}$ *of submodules of M,*

$$\bigcap_{\alpha\in\Lambda} M_\alpha = 0 \Rightarrow \bigcap_{\alpha\in\Lambda'} M_\alpha = 0$$

for some finite subset Λ' *of* Λ.

We show that finitely generated and finitely cogenerated modules can be characterized as modules that have certain chain conditions on their submodules.

Definition. *An R-module M is called* noetherian (artinian) *if for every ascending (descending) sequence of R-submodules of M,*

$$M_1 \subset M_2 \subset M_3 \subset \cdots \qquad (M_1 \supset M_2 \supset M_3 \supset \cdots),$$

there exists a positive integer k such that

$$M_k = M_{k+1} = M_{k+2} = \cdots.$$

If M is noetherian (artinian), then we also say that the ascending (descending) chain condition for submodules holds in M, or M has acc (dcc) on submodules, or simply that M has acc (dcc).

Because the ring of integers **Z** is a principal ideal ring, any ascending chain of ideals of **Z** is of the form

$$(n) \subset (n_1) \subset (n_2) \subset \cdots,$$

where $n, n_1, n_2, \ldots$ are in **Z**. Because $(n_i) \subseteq (n_{i+1})$ implies $n_{i+1} | n_i$, any ascending chain of ideals in **Z** starting with n can have at most n distinct terms. This shows that **Z** as a **Z**-module is noetherian. But **Z** as a **Z**-module has an infinite properly descending chain

$$(n) \supset (n^2) \supset (n^3) \supset \cdots,$$

showing that **Z** is not artinian as a **Z**-module.

Before we give more examples, we prove two theorems providing us with criteria for a module to be noetherian or artinian.

2.1 Theorem. *For an R-module M the following are equivalent:*

(i) M is noetherian.
(ii) Every submodule of M is finitely generated.
(iii) Every nonempty set S of submodules of M has a maximal element (that is, a submodule M_0 in S such that for any submodule N_0 in S with $N_0 \supset M_0$ we have $N_0 = M_0$).

Proof. (i) $\Rightarrow$ (ii). Let N be a submodule of M. Assume that N is not finitely generated. For any positive integer k let $a_1, \ldots, a_k \in N$. Then $N \neq (a_1, \ldots, a_k)$. Choose $a_{k+1} \in N$ such that $a_{k+1} \notin (a_1, \ldots, a_k)$. We then obtain an infinite properly ascending chain

$$(a_1) \subsetneq (a_1, a_2) \subsetneq \cdots \subsetneq (a_1, \ldots, a_k) \subsetneq (a_1, \ldots, a_{k+1}) \subsetneq \cdots$$

of submodules of M, which is a contradiction to the hypothesis. Hence, N is finitely generated.

(ii) $\Rightarrow$ (iii) Let N_0 be an element of S. If N_0 is not maximal, it is properly contained in a submodule $N_1 \in S$. If N_1 is not maximal, then N_1 is properly contained in a submodule $N_2 \in S$. In case S has no maximal elements, we obtain an infinite properly ascending chain of submodules $N_0 \subset N_1 \subset N_2 \subset \cdots$ of M. Let $N = \bigcup_i N_i$. N is also a submodule of M. For let $x, y \in \bigcup_i N_i$ and $r \in R$. Then $x \in N_u$, $y \in N_v$. Because either $N_u \subset N_v$ or $N_v \subset N_u$, both x and y lie in one submodule N_u or N_v, and, hence, $x - y$ and rx lie in the same submodule. This implies $x - y \in N$ and $rx \in N$, and, hence, N is a submodule of M. By (ii) N is finitely generated. So there exist elements $a_1, a_2, \ldots, a_n \in N$ such that $N = (a_1, a_2, \ldots, a_n)$. Now $a_1, a_2, \ldots, a_n$ belong to a finite number ($\leq n$) of submodules N_i, $i = 1, 2, \ldots$. Hence, there exists N_k such that all a_i, $1 \leq i \leq n$, lie in N_k. Because $N_k \subset N$ and N is the smallest submodule containing all a_i, $1 \leq i \leq n$, it follows that $N_k = N$. But then $N_k = N_{k+1} = \cdots$, a contradiction. Thus, S must have a maximal element.

(iii) $\Rightarrow$ (i) Suppose we have an ascending sequence of submodules of M,

$$M_1 \subset M_2 \subset M_3 \quad \cdots.$$

By (iii) the sequence $M_1, M_2, M_3, \ldots$ has a maximal element say M_k. But then $M_k = M_{k+1} = \cdots$. Hence, M is noetherian. □

The next theorem is dual to Theorem 2.1.

2.2 Theorem. *For an R-module M the following are equivalent:*

(i) M is artinian.
(ii) Every quotient module of M is finitely cogenerated.
(iii) Every nonempty set S of submodules of M has a minimal element (that is, a submodule M_0 in S such that for any submodule N_0 in S with $N_0 \subset M_0$, we have $N_0 = M_0$).

Proof. The proof is similar (indeed dual) to the proof of Theorem 2.1 and is thus left as an exercise. □

Definition. *A ring R is called a* left noetherian (artinian) *ring if R regarded as a left R-module is noetherian (artinian).*

Similarly, we define right noetherian (artinian) rings.

Throughout, unless otherwise stated, by a noetherian (artinian) ring we mean a left noetherian (artinian) ring.

In view of the importance of noetherian and artinian rings in themselves, we rewrite Theorems 2.1 and 2.2 for rings as follows:

2.3 Theorem. *Let R be a ring. Then the following are equivalent:*

(i) R is noetherian (artinian).

(ii) Let A be any left ideal of R. Then A (R/A) is finitely generated (cogenerated).

(iii) Every nonempty set S of left ideals of R has a maximal (minimal) element.

In particular, every principal left ideal ring is a noetherian ring.

2.4 Examples

(a) Let V be an n-dimensional vector space over a field F. Then V is both noetherian and artinian. For, if W is a proper subspace of V, then dim $W <$ dim $V = n$. Thus any properly ascending (or descending) chain of subspaces cannot have more than $n + 1$ terms.

(b) Let A be a finite-dimensional algebra with unity over a field F. Then A as a ring is both left and right noetherian as well as artinian. To see this, let $[A:F] = n$. If we observe that each left or right ideal is a subspace of A over F, it follows that any properly ascending (or descending) chain cannot contain more than $n + 1$ terms.

In particular, (i) if G is a finite group and F a field, then the group algebra $F(G)$ is both a noetherian and an artinian ring; (ii) the $m \times m$ matrix ring F_m over a field F is also a noetherian and artinian ring; (iii) the ring of upper (as well as lower) triangular matrices over a field F is both noetherian and artinian.

(c) Let $R = F[x]$ be a polynomial ring over a field F in x. Because $F[x]$ is a principal ideal domain, it follows by Theorem 2.3 that $F[x]$ is a noetherian ring. But $F[x]$ is not artinian, for there exists a properly descending chain of ideals in R, namely,

$$R \supset Rx \supset Rx^2 \supset \cdots.$$

However, every proper homomorphic image R/A, where A is a nonzero ideal in R, is artinian, because we know that R is a PID. Hence, $A = (p(x))$, $p(x) \in F[x]$. Let $p(x) = a_0 + a_1x + \cdots + a_nx^n$. Then $F[x]/(p(x))$ is a finite-dimensional algebra over F with a basis $\{1, \bar{x}, \ldots, \bar{x}^{n-1}\}$. Hence, by Example (b), $R/A = F[x]/(p(x))$ is an artinian ring.

(d) Let D_n be the $n \times n$ matrix ring over a division ring D. Then D_n is an n^2-dimensional vector space over D, and each left ideal as well as each right ideal of D_n is a subspace over D. Thus, any ascending or descending chain of left (as well as right) ideals cannot contain more than n^2+1 terms. Thus, D_n is both noetherian and artinian ring.

(e) Let p be a prime number, and let

$$R = \mathbf{Z}(p^\infty) = \left\{ \frac{m}{p^n} \in \mathbf{Q} \,\middle|\, 0 \le \frac{m}{p^n} < 1 \right\}$$

be the ring where addition is modulo positive integers, and multiplication is trivial; that is, $ab = 0$ for all $a, b \in R$. Then

(i) Each ideal in R is of the form

$$A_k = \left\{ \frac{1}{p^k}, \frac{2}{p^k}, \ldots, \frac{p^k - 1}{p^k}, 0 \right\},$$

where k is some positive integer.

(ii) R is artinian but not noetherian.

Solution. (i) Let $A \neq (0)$ be any ideal of R, and let k be the smallest positive integer such that for some positive integer m, $m/p^k \notin A$. Consider n/p^i, with $i \ge k$ and $(n, p) = 1$. We assert that $n/p^i \notin A$. Now $n/p^i \in A$ implies $np^{i-k}/p^i = n/p^k \in A$. Also, by choice of k, $1/p^{k-1} \in A$. Because $(n, p) = 1$, we can find integers a and b such that $an + bp = 1$. Then from n/p^k, $1/p^{k-1} \in A$, we have that na/p^k (reduced modulo whole numbers) and bp/p^k (reduced modulo whole numbers) lie in A. Hence, $1/p^k \in A$, a contradiction. Thus, no n/p^i, $i \ge k$, $(n, p) = 1$ can lie in A. Hence,

$$A = \left\{ \frac{1}{p^{k-1}}, \frac{2}{p^{k-1}}, \ldots, \frac{p^{k-1} - 1}{p^{k-1}}, 0 \right\}.$$

This ideal is denoted by A_{k-1}.

(iii) Because each ideal contains a finite number of elements, each descending chain of ideals must be finite. Hence, R is artinian.

Clearly, the chain

$$A_1 \subset A_2 \subset A_3 \subset \cdots$$

is an infinite properly ascending chain of left ideals, showing that R is not noetherian. Note that although each ideal $A \neq R$ is finite and, hence, finitely generated, R itself is not finitely generated.

2.5 Theorem. *Every submodule and every homomorphic image of a noetherian (artinian) module is noetherian (artinian).*

Proof. Follows at once from Theorem 2.1 (Theorem 2.2). □

2.6 Theorem. *Let M be an R-module, and let N be an R-submodule of M. Then M is noetherian (artinian) if and only if both N and M/N are noetherian (artinian).*

Proof. Let N and M/N be noetherian, and let K be any submodule of M. Then $(K+N)/N$ is a submodule of M/N, and, hence, it is finitely generated (Theorem 2.1). But then $(K+N)/N \simeq K/(N \cap K)$ implies $K/(N \cap K)$ is finitely generated, say

$$\frac{K}{N \cap K} = (\bar{x}_1) + \cdots + (\bar{x}_m), \qquad \bar{x}_i \in \frac{K}{N \cap K}.$$

Then

$$K = (x_1) + \cdots + (x_m) + N \cap K, \qquad x_i \in K.$$

Further, because N is noetherian, its submodule $N \cap K$ is finitely generated, say by $y_1, \ldots, y_n \in N \cap K$. This implies

$$K = (x_1) + \cdots + (x_m) + (y_1) + \cdots + (y_n).$$

Hence, K is finitely generated, so M is noetherian. The converse is Theorem 2.5. The proof for the artinian case is similar. □

An equivalent statement of Theorem 2.6 in the terminology of exact sequences is as follows.

Let $0 \to M_1 \to M \to M_2 \to 0$ be an exact sequence of R-modules. Then M is noetherian (artinian) if and only if both M_1 and M_2 are noetherian (artinian).

2.7 Theorem. *A subring of a noetherian (artinian) ring need not be noetherian (artinian).*

Proof. For the artinian case the ring of rational numbers $\mathbf{Q}$ is an artinian ring, but its subring $\mathbf{Z}$ is not an artinian ring.

For the noetherian case, the ring of 2×2 matrices over the rational numbers $\mathbf{Q}$ is a noetherian ring, but its subring $\left[\begin{smallmatrix} \mathbf{Z} & \mathbf{Q} \\ 0 & \mathbf{Q} \end{smallmatrix}\right]$ is not noetherian – that is, not left noetherian [see Example 2.15(e)]. □

2.8 Theorem. *Let R_i, $1 \le i \le n$, be a family of noetherian (artinian) rings each with a unity element. Then their direct sum $R = \oplus\sum_{i=1}^{n} R_i$ is again noetherian (artinian).*

Proof. We know that each left ideal A of R is of the form $A_1 \oplus \dots \oplus A_n$, where A_i are left ideals in R_i. So if a left ideal $B = B_1 \oplus \dots \oplus B_n$ of R is such that $A \subset B$, then it is clear that $A_i \subset B_i$, $1 \leq i \leq n$. Hence, any properly ascending (descending) chain of left ideals in R must be finite because each R_i is noetherian (artinian). □

2.9 Theorem. *If J is a nil left ideal in an artinian ring R, then J is nilpotent.*

Proof. Suppose $J^k \neq (0)$ for any positive integer k. Consider a family $\{J, J^2, J^3, \dots\}$. Because R is artinian, this family has a minimal element, say $B = J^m$. Then $B^2 = J^{2m} \subset J^m = B$ implies $B^2 = B$. Consider another family

$$\mathscr{F} = \{A \mid A \text{ is a left ideal contained in } B \text{ with } BA \neq (0)\}.$$

Then $\mathscr{F} \neq \varnothing$ because $B \in \mathscr{F}$. Let A be a minimal element in $\mathscr{F}$. Then $BA \neq (0)$. This implies there exists an element $a \in A$ such that $Ba \neq 0$. But $Ba \subset A$ and $B(Ba) = B^2a = Ba \neq 0$. Thus, $Ba \in \mathscr{F}$. Hence, by minimality of A, $Ba = A$. This gives that there exists an element $b \in B$ such that $ba = a$. This implies $b^i a = a$ for all positive integers i. But because b is a nilpotent element, this implies $a = 0$, a contradiction. Therefore, for some positive integer k, $J^k = (0)$. □

2.10 Lemma. *Let R be a noetherian ring. Then the sum of nilpotent ideals in R is a nilpotent ideal.*

Proof. Let $B = \Sigma_{i \in \Lambda} A_i$ be the sum of nilpotent ideals in R. Because R is noetherian (i.e., left noetherian), B is finitely generated as a left ideal. Suppose $B = (x_1, \dots, x_m)$. Then each x_i lies in the sum of finitely many A_i's. Hence, B is contained in the sum of a finite number of A_i's, say (after reindexing if necessary) $A_1, \dots, A_n$. Thus, $B = A_1 + \dots + A_n$. Then by Problem 1 of Section 5 in Chapter 10, B is nilpotent. □

Recall that if S is any nonempty subset of a ring R, then $l(S) = \{x \in R \mid xS = 0\}$ is a left ideal of R called the *left annihilator* of S in R.

2.11 Theorem. *Let R be a noetherian ring having no nonzero nilpotent ideals. Then R has no nonzero nil ideals.*

Proof. Let N be a nonzero nil ideal in R. Let $\mathscr{F} = \{l(n) \mid n \in N, n \neq 0\}$ be a family of left annihilator ideals. Because R is noetherian, $\mathscr{F}$ has a maximal member, say $l(n)$. Let $x \in R$. Then $nx \in N$, so there exists a smallest positive integer k such that $(nx)^k = 0$. Now, clearly, $l(n) \subset l((nx)^{k-1})$. Because $(nx)^{k-1} \neq 0$, $l((nx)^{k-1}) \in \mathscr{F}$. But then by maximality of $l(n)$, $l(n) = l((nx)^{k-1})$. Now

$$(nx)^k = 0 \Rightarrow nx \in l((nx)^{k-1}) = l(n) \Rightarrow nxn = 0.$$

Now $(RnR)^2 = RnRRnR = 0$. Therefore, by hypothesis, $RnR = 0$. If $1 \in R$, then $n = 0$, a contradiction. So in this case we are done. Otherwise, consider the ideal $(n) = nR + Rn + RnR + n\mathbf{Z}$ generated by n. Set $A = nR + Rn$. Because $nxn = 0$, for all $x \in R$, $A^2 = 0$. Thus, $(n) = A + n\mathbf{Z}$. But then if $n^k = 0$, we have $(A + n\mathbf{Z})^k = 0$. Therefore, by hypothesis, $A + n\mathbf{Z} = 0$, which gives $n = 0$, a contradiction. Hence, R has no nonzero nil ideals. □

Remark. Indeed, one can similarly show that R has no nonzero right or left nil ideals.

Next we show that a nil ideal in a noetherian ring is nilpotent.

2.12 Theorem. *Let N be a nil ideal in a noetherian ring R. Then N is nilpotent.*

Proof. Let T be the sum of nilpotent ideals in R. Then R/T has no nonzero nilpotent ideals, for if A/T is nilpotent, then $(A/T)^m = (0)$ implies $A^m/T = (0)$; so $A^m \subset T$. But since, by Lemma 2.10, T is nilpotent, there exists a positive integer k such that $(A^m)^k = (0)$. Hence, A itself is nilpotent, so $A \subset T$. This implies $A/T = (\bar{0})$.

Consider the nil ideal $(N + T)/T$ in R/T. By Theorem 2.11, $(N + T)/T = (0)$. This implies $N \subset T$, which is a nilpotent ideal. Hence, N is nilpotent. □

2.13 Remark. *If R is an artinian ring with identity, then it is known that R is noetherian.*

2.14 Theorem (Hilbert basis theorem). *Let R be a noetherian ring. Then the polynomial ring $R[x]$ is also a noetherian ring.*

Proof. Let $\mathscr{F}$ and $\mathscr{F}'$ be the families of left ideals of R and $R[x]$, respectively. Let n be a nonnegative integer. Define a mapping ϕ_n: $\mathscr{F}' \to \mathscr{F}$, where

$$\phi_n(I) = \{a \in R \mid \exists ax^n + bx^{n-1} + \cdots \in I, a \neq 0\} \cup \{0\}.$$

It is easy to verify that $\phi_n(I) \in \mathscr{F}$. We claim that if $I, J \in \mathscr{F}'$ with $I \subset J$ and

$\phi_n(I) = \phi_n(J)$ for all $n \geq 0$, then $I = J$. Let $0 \neq f(x) \in J$ be of degree m. Because $\phi_m(I) = \phi_m(J)$, there exists $g_m(x) \in I$ with leading coefficient the same as that of $f(x)$, and $f(x) - g_m(x)$ is either 0 or of degree at most $m-1$. Suppose $f(x) - g_m(x) \neq 0$. Because $f(x) - g_m(x) \in J$, we can similarly find $g_{m-1}(x) \in I$ such that $f(x) - g_m(x) - g_{m-1}(x) \in J$ and is either 0 or of degree at most $m-2$. Continuing like this, we arrive, after at most m steps, at

$$f(x) - g_m(x) - g_{m-1}(x) - \cdots - g_i(x) = 0,$$

where $g_m(x), g_{m-1}(x), \ldots \in I$. But then $f(x) \in I$, which proves that $I = J$, as claimed.

Let $A_1 \subset A_2 \subset A_3 \subset \cdots$ be an ascending sequence of left ideals of $R[x]$. Then for each nonnegative integer n,

$$\phi_n(A_1) \subset \phi_n(A_2) \subset \phi_n(A_3) \subset \cdots$$

is an ascending sequence of left ideals of R; hence, there exists a positive integer $k(n)$ such that

$$\phi_n(A_{k(n)}) = \phi_n(A_{k(n)+1}) = \cdots. \tag{1}$$

Further, because R is noetherian, the collection of left ideals $(\phi_n(A_i))$, $n \in \mathbf{N}$, $i \in \mathbf{N}$, has a maximal element, say $\phi_p(A_q)$ (Theorem 2.3). Then

$$\begin{aligned} \phi_p(A_q) &= \phi_n(A_q) && (\text{for all n} \geq p) \\ &= \phi_n(A_j) && (\text{for all n} \geq p, j \geq q). \end{aligned}$$

Therefore, we may choose $k(n) = q$ for all $n \geq p$ in (1). Moreover, if $s = k(1) \cdots k(p-1)q$, then $\phi_n(A_s) = \phi_n(A_{s+1}) = \cdots$ for all $n \in \mathbf{N}$. Hence, by the result proved in the first paragraph, $A_s = A_{s+1} = \cdots$. Therefore, $R[x]$ is noetherian. □

2.15 Examples

(a) If R is noetherian, then each ideal contains a finite product of prime ideals.

Solution. Suppose that the family $\mathcal{F}$ of ideals in R that do not contain any product of prime ideals is nonempty. Then by Theorem 2.3, $\mathcal{F}$ has a maximal element, say A. Because $A \in \mathcal{F}$, A is not a prime ideal. Hence, there exist ideals B and C of R such that $BC \subset A$, but $B \not\subset A$, $C \not\subset A$. Consider

$$(B + A)(C + A) \subset BC + BA + AC + A^2 \subset A.$$

Because, $B + A \supsetneq A$, and $C + A \supsetneq A$, both $B + A$ and $C + A$ contain a