(ii) It is straightforward to verify that the nontrivial right ideals

$$A = \begin{pmatrix} n_0\mathbf{Z} & \mathbf{Q} \\ 0 & 0 \end{pmatrix}, \qquad 0 \neq n_0 \in \mathbf{Z}, \tag{1}$$

and

$$A = \begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix}, \qquad K \text{ an additive subgroup of } \mathbf{Q}, \tag{2}$$

of R are also left ideals. Therefore, we have an example of a ring that is not commutative in which each right ideal is an ideal. It is interesting to note that each left ideal of R is not an ideal. (Consider

$$A = \left\{ \begin{pmatrix} n_0 m & ma \\ 0 & 0 \end{pmatrix} \middle| \, m \in \mathbf{Z} \right\},$$

where n_0 and a are fixed elements in $\mathbf{Z}$ and $\mathbf{Q}$, respectively.)

Problems

1. Let R be a commutative ring with unity. Suppose R has no nontrivial ideals. Prove that R is a field.
2. Prove the converse of Problem 1.
3. Generalize Problem 1 to noncommutative rings with unity having no nontrivial right ideals.
4. Find all ideals in $\mathbf{Z}$ and also in $\mathbf{Z}/(10)$.
5. Find all ideals in a polynomial ring $F[x]$ over a field F.
6. Find right ideals, left ideals, and ideals of a ring $R = \left[\begin{smallmatrix} \mathbf{Q} & \mathbf{Q} \\ 0 & 0 \end{smallmatrix}\right]$.
7. Show that every nonzero ideal in the ring of formal power series $F[[x]]$ in an indeterminate x over a field F is of the form (x^m) for some nonnegative integer m.
8. Let $L(R)$ be the set of all right (left) ideals in a ring R. Show that $L(R)$ is a modular lattice (see Problem 17, Section 1, Chapter 5). Give an example to show that the lattice $L(R)$ need not be distributive.

[A lattice is called distributive if $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ for all $a, b, c \in L$.]

2 Homomorphisms

Let R be a ring, and let I be an ideal in R. Let R/I be the quotient ring of R modulo I. Then there exists a natural mapping

$$\eta: R \to R/I$$

that sends $a \in R$ into $\bar{a} \in R/I$. This mapping preserves binary operations