Norwegian University of Science and Technology
Department of Mathematical Sciences

Contact during the exam:
Idun Reiten (99 24 45 39)

EXAM IN MA3201 RINGS AND MODULES

Friday December 11, 2009
Time: 09:00 – 13:00
Grades to be announced: Wednesday December 30, 2009
Permitted aids: None.
English

You should give a reason for all answers.

Problem 1

Let F be a field, $R = \left\{ \begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & 0 & e \end{pmatrix} ; a, b, c, d, e \in F \right\}$ and $I = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ b & 0 & 0 \\ d & 0 & 0 \end{pmatrix} ; b,\ d \in F \right\}$

a)

Show that R is a subring of the ring $M_3(F) = \begin{pmatrix} F & F & F \\ F & F & F \\ F & F & F \end{pmatrix}$ and that I is an ideal in R.

b) Show that the factor ring R/I is isomorphic to the ring $F \times F \times F$. Is R/I a semisimple ring?

c) Is R a left noetherian ring?

d)

$$\text{Let } I_1 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ d & 0 & 0 \end{pmatrix} ; d \in F \right\} \text{ and } I_2 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e \end{pmatrix} ; e \in F \right\}$$

Show that I_1 is a minimal left ideal, and that I_1 and I_2 (which is also a left ideal in R) are isomorphic as R-modules. Find a left ideal I_3 different from I_1 and I_2 such that I_1 and I_3 are isomorphic R-modules.

Problem 2

a) What is meant by the Smith normal form of an $m \times n$-matrix A over a principal ideal domain R? Find the Smith normal form of

$$A = \begin{pmatrix} 6 & 4 & 2 \\ 4 & 2 & 2 \\ 8 & 6 & 6 \end{pmatrix} \text{ over } \mathbb{Z}$$

b) Let $m(x) = (x-2)(x-1)^3$ be the minimal polynomial for a 6×6 - matrix A over $\mathbb{R}$. Find the possibilities for the invariant factors for $A - xI$ over $\mathbb{R}$, and write down the rational canonical form for A in one of the cases.

Problem 3

Let R be a semisimple ring and I an ideal in R. Show that the factor ring R/I is a semisimple ring.

Problem 4

Let A be a left ideal in a ring R, and assume that $A = Aa$ for some $a \neq 0$ in A.

(i) Show that there is some $e \in A$ where $ea \neq 0$ and $(e^2 - e)a = 0$.

(ii) Let $B = \{x \in A; xa = 0\}$. Show that B is a left ideal in R.

(iii) Assume further that the left ideal A is a minimal left ideal. Show that the element e from (i) is then an idempotent element.

(You can use (i) to show (ii) even if you do not show (i), and you can use (i) and (ii) to show (iii)).

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Contact during the exam:
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EXAM IN MA3201 RINGS AND MODULES

Thursday Desember 4, 2008
Time: 09.00 – 13:00
Sensurdato: Monday, 5. January 2009
Permitted aids: None.
English

You must give arguments for all your answers.

$\mathbb{R}$ denotes the real numbers and $\mathbb{Z}$ denotes the integers.

Problem 1

Let F be a field and

$$R = \begin{pmatrix} F & 0 & 0 \\ F & F & 0 \\ F & 0 & F \end{pmatrix}$$

a) Show that R is a ring and that $I = \begin{pmatrix} 0 & 0 & 0 \\ F & 0 & 0 \\ F & 0 & 0 \end{pmatrix}$ is an ideal in R.

b) Show that the factor ring R/I is isomorphic to the ring $F \times F \times F$. Is R/I a semisimple ring?

c) Show that $I_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ F & 0 & 0 \end{pmatrix}$ and $I_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & F & 0 \\ 0 & 0 & 0 \end{pmatrix}$

are minimal left ideals in R, and that I_1 and I_2 are not isomorphic R-modules.

Problem 2

a) Let R be a ring and M a noetherian R-module. Let N be a submodule of M. Show that the factor module M/N is a noetherian R-module.

b) Show that the ring of integers $\mathbb{Z}$ is not an artinian ring.

Problem 3

Let R be a ring and I a left ideal in R. Assume there is a left ideal J in R such that $R = I \bigoplus J$. Show that $I = Re$, where e is an idempotent.

Let F be a field and $I = \begin{pmatrix} F & 0 \\ F & 0 \end{pmatrix}$ a left ideal in the ring $\begin{pmatrix} F & 0 \\ F & F \end{pmatrix}$. Find an idempotent e in R such that $I = Re$, and a left ideal J in R such that $R = I \bigoplus J$.

Problem 4

Find a nonzero nilpotent ideal in the ring $\mathbb{Z}/(4)$. For which $n \geq 1$ is the ring

$$\begin{pmatrix} Z/(2^n) & Z/(2^n) \\ Z/(2^n) & Z/(2^n) \end{pmatrix} \text{ semisimple?}$$

Problem 5

a) Find Smith normal form over $\mathbb{Z}$ of the matrix

$$A = \begin{pmatrix} 0 & 2 & -1 \\ -3 & 8 & 3 \\ 2 & -4 & -1 \end{pmatrix}$$

b) Let A be a 6×6 matrix over $\mathbb{R}$ with minimal polynomial $m(x) = (x^2+1)(x-2)(x-1)$. Find all possibilities for the non-unit monic invariant factors for the matrix $A - xI_6$. In each case, find the corresponding rational canonical form for A.

c) Let V be a vector space over $\mathbb{R}$ of dimension 4, and let $T : V \to V$ be a linear transformation. View V(with T) as $\mathbb{R}[x]$-module in the usual way. Assume that $\{v_1, v_2, Tv_2, T^2v_2\}$ is a basis for the vector space V, for some v_1, v_2 in V, and that $Tv_1 = v_1$ and $T^3v_2 = 4T^2(v_2) - 5T(v_2) + 2v_2$. Find $f_1(x)$ and $f_2(x)$ in $\mathbb{R}[x]$, with $f_1(x)|f_2(x)$ such that $V \simeq \mathbb{R}[x]/(f_1(x)) \bigoplus \mathbb{R}[x]/(f_2(x))$ as $\mathbb{R}[x]$-modules.

Norwegian University of Science and Technology
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Contact during midterm exam:
Idun Reiten (73 59 17 42)

MIDTERM EXAM IN MA3201 RINGS AND MODULES

Friday, October 5, 2007
Time: 12:15 – 14:00, F3
Permitted aids: None.
English

You must give arguments for all your answers.

Problem 1

Let S be the continuous functions from $[0,1]$ to $\mathbf{R}$ (real numbers), where $(f+g)(x) = f(x)+g(x)$ and $(fg)(x) = f(x)g(x)$, for f, g i S and $x \in [0,1]$. Then S is a commutative ring. (Should not be shown.)

a) Show that $I = \{f \in S; f(1/2) = 0\}$ is an ideal in S

b) Find some $f \neq 0$ in S which is a zero divisor.

c) Show that the factor ring S/I is isomorphic to the ring $\mathbf{R}$.

Problem 2

Let F be a field and R the ring $\left(\begin{smallmatrix} F & F \\ F & F \end{smallmatrix}\right)$.

a) Show that $I = \left(\begin{smallmatrix} F & 0 \\ F & 0 \end{smallmatrix}\right)$ is a minimal left ideal in $\mathbf{R}$.

b) Show that the sum $R = \left(\begin{smallmatrix} F & 0 \\ F & 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & F \\ 0 & F \end{smallmatrix}\right)$ of left ideals is a direct sum.

c) Is R a semisimple R-module? (semisimple = completely reducible).

Problem 3

Let $\mathbf{Z}$ be the ring of integers.

a) Show that $\mathbf{Z}$ has no simple submodules.

b) Is $\mathbf{Z}$ a semisimple $\mathbf{Z}$-module?

c) Is $\mathbf{Z}$ a free $\mathbf{Z}$-module?

Problem 4

Find all left ideals in the ring $R = \begin{pmatrix} \mathbf{Z}_2 & 0 \\ \mathbf{Z}_2 & \mathbf{Z}_2 \end{pmatrix}$

Norwegian University of Science and Technology
Department of Mathematical Sciences

Contact during exam: Øyvind Solberg
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EXAM IN RINGS AND MODULES (MA3201)

English
Friday 15th December 2006
Time: 09:00–13:00
Permitted aids: None

Grades: 15.01.2007.

Problem 1 Let A be the 3×3 matrix

$$\begin{pmatrix} 1 & 2 & -4 \\ 1 & 2 & 2 \\ -1 & 1 & 1 \end{pmatrix}$$

over $\mathbb{C}$, the complex numbers.

a) Find the Smith normal form of the matrix $A - xI_3$ over the ring $\mathbb{C}[x]$, where $\mathbb{C}[x]$ is the polynomial ring in one variable x over $\mathbb{C}$ and I_3 is the 3×3 identity matrix.

b) Find the rational canonical form of the matrix A over $\mathbb{C}$.

c) Find the Jordan canonical form of the matrix A over $\mathbb{C}$.

Problem 2 Let R and S be two rings. An abelian group M is called a S-R-bimodule if M is a left S-module and a right R-module, such that

$$s(mr) = (sm)r$$

for all s in S, for all r in R and for all m in M. Let

$$\Lambda = \begin{pmatrix} R & 0 \\ M & S \end{pmatrix}$$

where M is a S-R-bimodule different from (0). Let $\begin{pmatrix} r & 0 \\ m & s \end{pmatrix}$ and $\begin{pmatrix} r' & 0 \\ m' & s' \end{pmatrix}$ be two elements in Λ. The set Λ becomes an abelian group under the binary operation, $+$, given by

$$\begin{pmatrix} r & 0 \\ m & s \end{pmatrix} + \begin{pmatrix} r' & 0 \\ m' & s' \end{pmatrix} = \begin{pmatrix} r+r' & 0 \\ m+m' & s+s' \end{pmatrix}.$$

Define a binary operation, $\cdot$, on Λ by letting

$$\begin{pmatrix} r & 0 \\ m & s \end{pmatrix} \cdot \begin{pmatrix} r' & 0 \\ m' & s' \end{pmatrix} = \begin{pmatrix} rr' & 0 \\ mr'+sm' & ss' \end{pmatrix}.$$

a) Show that Λ is a ring with 1, when addition, $+$, and multiplication, $\cdot$, is defined as above.

b) Find

(i) an idempotent element different from 0 and 1 in Λ,

(ii) a nilpotent element different from 0 i Λ.

c) Let $I = \{\begin{pmatrix} 0 & 0 \\ m & 0 \end{pmatrix} \mid m \in M\}$. Show that I is a twosided ideal in Λ. Show that $\Lambda/I \simeq R \oplus S$ as rings.

Problem 3 Let k be a field. The map $\varphi\colon k[x]/(x^2) \to k$ given by

$$\varphi(f(x) + (x^2)) = f(0)$$

is a homomorphism of rings. Let $R = k$ and $S = k[x]/(x^2)$.

a) Let M be a left R-module. Show that M becomes a left S-module by defining

$$s \cdot m = \varphi(s)m$$

for all s in S and for all m in M.

b) Let $M = k^2 = \{(a, b) \mid a, b \in k\}$. Then is M a left k-module by letting

$$\alpha(a, b) = (\alpha a, \alpha b)$$

and a right k-module by letting

$$(a, b)\alpha = (a\alpha, b\alpha)$$

for all α in k and for all (a, b) in M. With these module structures M becomes a k-k-bimodule (Do not need to show this). By a) we have that the left k-module M is a left S-module by letting $(f(x) + (x^2)) \cdot m = \varphi(f(x) + (x^2))m$. Show that M is a S-R-bimodule, when $R = k$ and $S = k[x]/(x^2)$.

c) Now let $\Lambda = \left(\begin{smallmatrix} R & 0 \\ M & S \end{smallmatrix}\right)$, where M is as in b), and Λ is a ring as given in Problem 2. Show that Λ is an algebra over k. What is $\dim_k \Lambda$? Decide if Λ is

(i) a left artinian ring,

(ii) a left noetherian ring,

(iii) a semisimple ring.

d) Let J be the left ideal $\{\left(\begin{smallmatrix} 0 & 0 \\ (0,a) & bx+(x^2) \end{smallmatrix}\right) \mid a, b \in k\}$. Consider the left Λ-module $X = \Lambda/J$. Show that $f\colon X \to X$ given by

$$f(\lambda + J) = \lambda \left(\begin{smallmatrix} 0 & 0 \\ (0,0) & 1+(x^2) \end{smallmatrix}\right) + J$$

is a Λ-homomorphism. Find the image $\operatorname{Im} f$ of f. Show that $X = \operatorname{Im} f \oplus Y$ for a submodule Y of X.

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Contact during exam: Øyvind Solberg/Petter Andreas Bergh
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Exam in course MA3201 Rings and modules

English
Wednesday November 30, 2005
Time: 09.00-13.00

Permitted aids: none

Grades: 21.12.2005.

Problem 1 Let q be a fixed non-zero element in $\mathbb{C}$, the set of complex numbers. Define the subset R_q of the ring of 4×4-matrices over $\mathbb{C}$ by

$$R_q = \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ b & a & 0 & 0 \\ c & 0 & a & 0 \\ d & c & -qb & a \end{pmatrix} \mid a, b, c, d \in \mathbb{C} \right\}.$$

a) Show that R_q is a ring.

b) For which q in $\mathbb{C}$ is R_q a commutative ring?

c) For a given element α in $\mathbb{C}$ define the subset

$$I_\alpha = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ b & 0 & 0 & 0 \\ \alpha b & 0 & 0 & 0 \\ d & \alpha b & -qb & 0 \end{pmatrix} \mid b, d \in \mathbb{C} \right\}$$

of R_q. Show that I_α is a left ideal in R_q for all α in $\mathbb{C}$.

d) Show that each of the left ideals I_α is generated by one element as a left ideal. Show that $I_\alpha \simeq R/I_{\alpha q}$ as left R-modules.

Problem 2 Let $\mathbb{Q}$ be the field of rational numbers, and let a and b in $\mathbb{Q}$ be different elements. Find all possible rational canonical forms for 4×4-matrices over $\mathbb{Q}$ having

$$(x+a)^2(x+b)$$

as a minimal polynomial.

Problem 3 Let $\mathbb{C}$ be the field of complex numbers and $\mathbb{C}[x]$ the polynomial ring over $\mathbb{C}$ in one variable x. Let $\alpha \in \mathbb{C}$ be a complex number.

a) Show that the map $\varphi_\alpha \colon \mathbb{C}[x] \to \mathbb{C}$ defined by $\varphi_\alpha(f(x)) = f(\alpha)$ is a surjective ring homomorphism, and use this to show that the ideal generated by $x - \alpha$ is a maximal ideal in $\mathbb{C}[x]$.

b) For which $n \geq 1$ is the ring

$$\begin{pmatrix} \frac{\mathbb{C}[x]}{((x-\alpha)^n)} & \frac{\mathbb{C}[x]}{((x-\alpha)^n)} \\ \frac{\mathbb{C}[x]}{((x-\alpha)^n)} & \frac{\mathbb{C}[x]}{((x-\alpha)^n)} \end{pmatrix}$$

semisimple?

Problem 4 Let R be a ring, and let M be a Noetherian left R-module. Show that any surjective R-homomorphism $f \colon M \to M$ is an isomorphism. (Hint: Consider the chain $\operatorname{Ker} f \subseteq \operatorname{Ker}(f^2) \subseteq \operatorname{Ker}(f^3) \subseteq \cdots$ of submodules of M).

Norwegian University of Science and Technology
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Faglig kontakt under eksamen:
Petter Andreas Berg (73 59 04 83)

EXAM IN RINGS AND MODULES (MA3201)

Thursday, 9th December 2004
Time: 09:00 – 13:00
Grades to be announced: Thursday, 6th January 2005
Permitted aids: None.

Problem 1 Let

$$R = \left\{ \begin{pmatrix} a & 0 & 0 \\ b & a & 0 \\ c & d & e \end{pmatrix} \mid a, b, c, d, e \in \mathbb{C} \right\}.$$

a) Show that R is a ring under the usual addition and multiplication of matrices.

b) Let

$$I = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ b & 0 & 0 \\ c & d & 0 \end{pmatrix} \mid b, c, d \in \mathbb{C} \right\}.$$

Show that I is a two-sided ideal in R, and that I is nilpotent.

c) Show that R/I and $\mathbb{C} \oplus \mathbb{C}$ are isomorphic rings. Is R/I a semisimple ring?

d) How can the two-sided ideals in the ring R/I be described in terms of two-sided ideals in R? Find two maximal two-sided ideals in R.

Problem 2

a) Let $\varphi\colon R \to S$ be a homomorphism of rings. Show that any left S-module M becomes a left R-module by defining

$$r \cdot m = \varphi(r)m$$

for all r in R and m in M.

Recall the following: Let F be a field. Suppose A is an algebra over F; that is, there is a map $F \times A \to A$, written $(\alpha, r) \mapsto \alpha \cdot r$, such that A is a vector space over F and

$$\alpha \cdot (rr') = (\alpha \cdot r)r' = r(\alpha \cdot r')$$

for all α in F, and all r and r' in A.

Assume that $0 \neq 1_A$ in A, where 1_A is the identity in A.

b) Show that $\psi\colon F \to A$ given by $\psi(\alpha) = \alpha \cdot 1_A$, is a homomorphism of rings with $\operatorname{Im}\psi \subseteq Z(A)$. Here

$$Z(A) = \{z \in A \mid za = az \text{ for all } a \in A\}.$$

Also, show that ψ is injective.

c) Suppose that A is a finite dimensional algebra over F; that is, $\dim_F A$ is finite. Show that A is both left artinian and left noetherian.

Let M be a finitely generated left A-module. Show that M is both an artinian and a noetherian A-module.

Problem 3 Let V be a vector space over a field F with $\dim_F V = n < \infty$. Let $T\colon V \to V$ be a non-zero linear transformation. Then V becomes an $F[x]$-module by letting

$$x^i \cdot v = T^i(v)$$

for all v in V and $i \geq 0$. It is not necessary to prove this.

a) Let $\operatorname{Ann}_{F[x]} V = \{g(x) \in F[x] \mid g(x) \cdot v = 0 \text{ for all } v \in V\}$. Show that $\operatorname{Ann}_{F[x]} V$ is an ideal in $F[x]$.

Let $f(x)$ be the minimal polynomial of T. Show that $\operatorname{Ann}_{F[x]} V = (f(x))$.

b) Suppose that T is a non-zero nilpotent linear transformation; that is, $T^l = 0$ for some positive integer l. Show that the minimal polynomial $f(x)$ of T is equal to x^m for some integer m with $0 < m \leq n$.

c) Suppose also here that T is a non-zero nilpotent linear transformation. What is the smallest possible dimension of the kernel of T? And what is the largest possible dimension of the kernel of T?