



Scientific contact during the exam:
Tore Forbregd (73 59 16 25)

MA3201 Rings and modules

Monday 13th December 2010

Time: 09:00–13:00

Permitted aids: Simple calculator

Problem 1 Let $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 0 & 4 & 2 \\ 6 & 1 & -18 & -9 \\ -12 & 0 & 45 & 22 \end{pmatrix}$ in the full matrix ring $M_4(\mathbb{R})$, where \mathbb{R} denotes the real numbers.

a) Find the Smith normal form of the matrix

$$A - xI_4 = \begin{pmatrix} 1-x & 0 & 0 & 0 \\ -2 & -x & 4 & 2 \\ 6 & 1 & -18-x & -9 \\ -12 & 0 & 45 & 22-x \end{pmatrix}$$

over $\mathbb{R}[x]$, where I_4 denotes identity matrix in $M_4(\mathbb{R})$.

b) Compute the rational canonical form of A .

c) Compute the Jordan canonical form of A .

Problem 2 Let $A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ in the full matrix ring $M_3(\mathbb{Z}_2)$, where \mathbb{Z}_2 is the field with two elements.

Define $\psi: \mathbb{Z}_2[x] \rightarrow M_3(\mathbb{Z}_2)$ by

$$\psi(f(x)) = a_0I_3 + a_1A + a_2A^2 + \cdots + a_mA^m,$$

when $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_mx^m$ in $\mathbb{Z}_2[x]$ and I_3 denotes the identity matrix in $M_3(\mathbb{Z}_2)$.

- a) Show that ψ is a homomorphism of rings.
- b) Find the kernel $\text{Ker } \psi$ of ψ , and show that the image of ψ , denoted by $\text{Im } \psi$, is a subring of $M_3(\mathbb{Z}_2)$ and a field with 8 elements.
- c) Let $F = \text{Im } \psi$. Why is $M_3(\mathbb{Z}_2)$ not an algebra over F , when the action of the subring F on $M_3(\mathbb{Z}_2)$ is the natural one? Find a field k such that $M_3(\mathbb{Z}_2)$ is an algebra over k , and compute the dimension of $M_3(\mathbb{Z}_2)$ as a vector space over k .

Problem 3 Let R be the subset $\left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ c & 0 & b & 0 \\ 0 & d & 0 & a \end{pmatrix} \mid a, b, c, d \in \mathbb{C} \right\}$ of the full 4×4 -matrix ring $M_4(\mathbb{C})$ over the complex numbers \mathbb{C} .

- a) Show that R is a ring with 1.
- b) Let I be the subset $\left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ c & 0 & 0 & 0 \\ 0 & d & 0 & 0 \end{pmatrix} \mid c, d \in \mathbb{C} \right\}$ of R . Show that I is a two-sided ideal in R , and that $R/I \simeq \mathbb{C} \oplus \mathbb{C}$. Is R a semisimple ring?
- c)
 - (i) Show that I is a nilpotent ideal in R .
 - (ii) Show that any two-sided prime ideal A in R contains any nilpotent two-sided ideal J , i.e. $J \subseteq A$ for all nilpotent two-sided ideal J in R .
 - (iii) Find two different maximal two-sided ideals in R .
- d) Let M be a left R -module, and let

$$IM = \left\{ \sum_{i=1}^n a_i x_i \mid a_i \in I, x_i \in M, \text{ for all } i = 1, 2, \dots, n, n \text{ a positive integer} \right\}.$$

- (i) Show that IM is a submodule of M .
- (ii) Let I be as in (b). The factor M/IM becomes a left module over R via

$$r \cdot (m + IM) = rm + IM,$$

for r in R and m in M . Assume that M/IM has $\{\overline{m_1}, \overline{m_2}, \dots, \overline{m_t}\}$ as generators as a left R -module. Let $\{m_i\}_{i=1}^t$ be elements in M such that $\overline{m_i} = m_i + IM$ in M/IM . Show that $\{m_i\}_{i=1}^t$ generates M as left R -module.



Faglig kontakt under eksamen: Øyvind Solberg
Telefon: 73 59 17 48

EKSAMEN I RINGER OG MODULER (MA3201)

Norsk

Fredag 9. juni 2011

Tid: 09:00–13:00

Hjelpermidler: Ingen

Sensurdato: 20.06.2011.

Oppgave 1 La A være 3×3 -matrisen

$$\begin{pmatrix} 1 & 2 & -4 \\ 1 & 2 & 2 \\ -1 & 1 & 1 \end{pmatrix}$$

over \mathbb{C} , de komplekse tall.

- Finn Smith normal form av matrisen $A - xI_3$ over ringen $\mathbb{C}[x]$, hvor $\mathbb{C}[x]$ er polynomringen i en variabel x over \mathbb{C} og I_3 er 3×3 -identitetsmatrisen.
- Finn den rasjonale kanoniske formen til matrisen A over \mathbb{C} .
- Finn den Jordan-kanoniske formen til matrisen A over \mathbb{C} .

Oppgave 2 La $A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ i den fulle matriseringen $M_3(\mathbb{Z}_2)$, der \mathbb{Z}_2 er kroppen med to elementer.

Definer $\psi: \mathbb{Z}_2[x] \rightarrow M_3(\mathbb{Z}_2)$ ved at for $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_mx^m$ i $\mathbb{Z}_2[x]$, vi har

$$\psi(f(x)) = a_0I_3 + a_1A + a_2A^2 + \cdots + a_mA^m,$$

der I_3 er identitetsmatrisen i $M_3(\mathbb{Z}_2)$.

- a) Vis at ψ er en homomorf av ringer.
- b) Finn $\text{Ker } \psi$, og vis at bildet til ψ , som vi betegner $\text{Im } \psi$, er en underring av $M_3(\mathbb{Z}_2)$ og en kropp med 8 elementer.
- c) La $F = \text{Im } \psi$. Hvorfor er ikke $M_3(\mathbb{Z}_2)$ en algebra over F , når virkningen av underringen F på $M_3(\mathbb{Z}_2)$ er den naturlige? Finn en kropp k slik at $M_3(\mathbb{Z}_2)$ er en algebra over k , og bestem dimensjonen til $M_3(\mathbb{Z}_2)$ som vektorrom over k .

Oppgave 3 La V være et vektorrom over en kropp F der $\dim_F V = n < \infty$. La $T: V \rightarrow V$ være en ikke-null lineær transformasjon. Da blir V en $F[x]$ -modul ved å definere

$$x^i \cdot v = T^i(v)$$

for alle $v \in V$ og $i \geq 0$. Det er ikke nødvendig å vise dette.

- a) La $\text{Ann}_{F[x]} V = \{g(x) \in F[x] \mid g(x) \cdot v = 0 \text{ for alle } v \in V\}$. Vis at $\text{Ann}_{F[x]} V$ er et ideal i $F[x]$.
La $f(x)$ være minimalpolynomet til T . Vis at $\text{Ann}_{F[x]} V = (f(x))$.
- b) Anta at T er en ikke-null nilpotent lineær transformasjon, dvs. $T^l = 0$ for et positivt helt tall l . Vis at minimalpolynomet $f(x)$ til T er lik x^m for et helt tall m med $0 < m \leq n$.
- c) Anta også i dette punktet at $T: V \rightarrow V$ en ikke-null nilpotent linear transformasjon.
Hva er den minste mulige dimensjonen av kjernen til T ? Og hva er den størst mulige dimensjonen til kjernen til T ?



Scientific contact during the exam:

David Jørgensen (73 59 34 64)

MA3201 Rings and modules

Thursday 1st December 2011

Time: 09:00–13:00

Permitted aids: Simple calculator

Problem 1 Let $A = \begin{pmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$ in the full matrix ring $M_4(\mathbb{Q})$, where \mathbb{Q} denotes the rational numbers.

a) Find the Smith normal form of the matrix

$$A - xI_4 = \begin{pmatrix} 2-x & 0 & -1 & 0 \\ 0 & 2-x & 0 & -1 \\ 0 & 0 & 2-x & 0 \\ 0 & 0 & 0 & 2-x \end{pmatrix}$$

over $\mathbb{Q}[x]$, where I_4 denotes identity matrix in $M_4(\mathbb{Q})$.

b) Compute the rational canonical form of A .

c) Compute the Jordan canonical form of A .

Problem 2 Let F be a field, and $R = \begin{pmatrix} F & F & F \\ 0 & F & 0 \\ 0 & 0 & F \end{pmatrix}$.

a) Show that R is a subring of the full matrix ring $M_3(F)$.

- b) Show that both $I_1 = \begin{pmatrix} 0 & F & 0 \\ 0 & F & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $I_2 = \begin{pmatrix} 0 & F & F \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ are (two-sided) ideals of R .
- c) We define a ring to be *semisimple* if it is a finite direct sum of matrix rings over division rings. Give 3 equivalent conditions for a ring R to be semisimple.
- d) Is R/I_1 semisimple? Is R/I_2 semisimple? Why or why not?

Problem 3 Let R be a ring (with unity 1). Show that every proper left ideal I is contained in a maximal left ideal of R .

Problem 4 Define a mapping $\iota : \mathbb{C}[t] \rightarrow \mathbb{C}[x, y]$ by $\iota(f(t)) = f(x + y) \in C[x, y]$, for $f(t) \in \mathbb{C}[t]$.

- a) Show that ι is a homomorphism of rings.
- b) Show that if $\psi : R \rightarrow S$ is a homomorphism of rings, and M is a left S -module, then M is also a left R -module via the action $r \cdot x = \psi(r)x$, for $r \in R$ and $x \in M$.
- c) State the Decomposition Theorem for finitely generated modules over a PID.
- d) Consider the ring homomorphism

$$\varphi : \mathbb{C}[t] \rightarrow \mathbb{C}[x, y]/(xy)$$

defined as the composition $\varphi = \pi \circ \iota$, where $\pi : \mathbb{C}[x, y] \rightarrow \mathbb{C}[x, y]/(xy)$ is the natural projection (φ is also injective, but you do not need to show this). According to Part (b), $\mathbb{C}[x, y]/(xy)$ is itself a module over the ring $\mathbb{C}[t]$ via φ . Show that it is finitely generated as a $\mathbb{C}[t]$ -module, and write its decomposition, up to isomorphism, according to the Decomposition Theorem for finitely generated modules over a PID.



Scientific contact during the exam:
Aslak Bakke Buan 73 55 02 89/40 84 04 68

Exam in MA3201:
Rings and modules

English
December 15. 2012
Tid: 0900-1300

Permitted aids:
simple calculator

All answers should be justified and properly explained.

Problem 1

Find Smith normal form over the integers \mathbb{Z} for the matrix $\begin{bmatrix} 2 & 4 & 2 \\ 2 & 4 & 6 \\ 3 & 5 & 7 \end{bmatrix}$.

Problem 2

Consider the ring $R = \left\{ \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\}$, where \mathbb{R} denotes the real numbers, and the subset $I = \left\{ \begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix} \mid b \in \mathbb{R} \right\}$.

- a) Show that I is an ideal in R .

Is the ring R commutative, artinian, noetherian, semisimple? Is the ring R/I commutative, artinian, noetherian, semisimple?

- b) Find two maximal ideals m_1, m_2 in R , such that the intersection $m_1 \cap m_2 = I$.
- c) Show that there are no other maximal ideals in R .
- d) Find two simple R -modules which are not isomorphic.

Problem 3

- a) For any ring R and any ideal I in R , show that the left modules over R/I are exactly the left R -modules M such that $IM = 0$.
- b) Let $R = F[x]$ for a field F and let $I = (x^2)$. Show that for an R/I -module M , the following three statements are equivalent:
 - M is finitely generated as an R -module.
 - M is finitely generated as an R/I -module.
 - M is finite dimensional as an F -vector space ($=F$ -module).
- c) Classify all finitely generated modules over $F[x]/(x^2)$ (up to isomorphism).

Problem 4

Let R be a ring, and let M be a left R -module.

- a) Show that if M is noetherian, then all submodules of M are finitely generated.
- b) Show that if M is both noetherian and artinian, then there is a finite sequence of submodules

$$M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_{n-1} \supseteq M_n = 0$$

such that M_i/M_{i+1} is a simple R -module, for $i = 0, \dots, n - 1$.

Give an example to show that such a finite sequence does not necessarily exist if M is only noetherian (and not artinian).



Faglig kontakt:
Aslak Bakke Buan 73 55 02 89/40 84 04 68

Eksamens i MA3201:

Ringer og moduler

Bokmål

7. august, 2013

Tid: 0900-1300

Tillatte hjelpeemidler:
enkel kalkulator

Alle svar må begrunnes og forklares.

Oppgave 1

Finn Smith normalform over heltallene \mathbb{Z} for matrisen $\begin{bmatrix} 4 & 2 & 4 \\ 12 & -6 & 6 \\ -16 & 10 & -4 \end{bmatrix}$.

Oppgave 2 La A være en 4×4 -matrise over \mathbb{R} med minimalpolynom $m_A(x) = (x - 3)^2$. Hva er de mulige rasjonalkanoninske formene A kan ha?

Oppgave 3 La G være en gruppe av orden 2 og la \mathbb{Z}_2 være kroppen med 2 elementer. Er grupperingen $\mathbb{Z}_2 G$ semisimpel?

Oppgave 4 La $F = \mathbb{Z}_2$, og betrakt mengden av alle matriser

$$R = \left\{ \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & 0 & e \end{bmatrix} \mid a, b, c, d, e \in \mathbb{Z}_2 \right\}$$

- a) Vis at R er en underring av ringen $M_3(\mathbb{Z}_2)$ av alle 3×3 -matriser over \mathbb{Z}_2 . Er R en kommutativ ring? Er R en semisimpel ring? Er R en venstreartinsk ring?
- b) Finn to ekte, ikke-trivuelle, tosidige idealer I og J , slik at R/I er semisimpel og R/J ikke er semisimpel.
- c) Det finnes 6 forskjellige venstreidealere J_1, \dots, J_6 i R , med $\dim_{\mathbb{Z}_2} J_i = 1$. Finn disse venstreidealene, og vis at de gir opphav til to simple ikke-isomorfe venstre R -moduler.

Oppgave 5 La R være en ring, og la $\phi: S \rightarrow R$ være en ringhomomorf.

- a) Hva betyr det at en venstre R -modul er *endeliggenerert*?
- b)
 - (i) Forklar på hvilken måte hver venstre R -modul også er en venstre S -modul.
 - (ii) Anta ϕ er surjektiv (på). Vis at en endeliggenerert venstre R -modul også er en endeliggenerert venstre S -modul.
- c)
 - (i) Skriv ned Zorns lemma.
 - (ii) Bevis at hver endeliggenererte venstre R -modul har en maksimal undermodul.