Exercises 3, MA3150 ANT

A II.1 (a) By partial summation,

$$\sum_{n\leq x} \frac{1}{n^s} = \frac{[x]}{x^s} + s\int_1^x \frac{[t]}{t^{s+1}}\,dt.$$

When $\sigma > 1$, we get by letting $x \to \infty$;

$$\zeta(s) = s\int_1^\infty \frac{[x]}{x^{s+1}}\,dx$$

(b) By partial summation,

$$\sum_{p\leq x} \frac{1}{p^s} = \frac{\pi(x)}{x^s} + s\int_1^x \frac{\pi(t)}{t^{s+1}}\,dt.$$

When $\sigma > 1$, we get by letting $x \to \infty$;

$$\sum_p \frac{1}{p^s} = s\int_1^\infty \frac{\pi(x)}{x^{s+1}}\,dx.$$

(c), (d) are done in exactly the same way.

A II.3 Noting that

$$\int_x^\infty \frac{[t]}{t^{s+1}}\,dt = \frac{x^{-s+1}}{1-s} + \int_x^\infty \frac{[t]-t}{t^{s+1}}\,dt$$

$$= \frac{x^{-s+1}}{1-s} + O(x^{-\sigma}),$$

we see from (a) above that

$$\sum_{n \le x} \frac{1}{n^s} = \frac{x^{1-s}}{1-s} + \zeta(s) + O(x^{-\sigma}),$$

This formula solves both (a) and (b).

A.13.2. We recognize that

$$A(x) = \sum_{n \le x} \frac{\Lambda(n)}{\log n}.$$

As shown by Davenport,

$$A(x) = \pi(x) + \tfrac{1}{2}\pi(x^{\frac{1}{2}}) + \tfrac{1}{3}\pi(x^{\frac{1}{3}}) + \cdots + \frac{1}{2[\log x]}\pi\left(x^{\frac{1}{2[\log x]}}\right)$$

$$\le \pi(x) + \tfrac{1}{2}\sqrt{x} + x^{\frac{1}{3}}\, 2\log\log x$$

$$= \pi(x) + O(\sqrt{x}),$$

A.13.3. (a) This is a direct consequence of Perron's formula and our expression for A(x) above.

(b) This is a direct consequence of A13.2.

A 13.4. $M(x) := \sum_{n \le x} \mu(n)$.

Suppose $M(x) = x^{\Theta}$, $x \geq 1$, and $\frac{1}{2} < \Theta < 1$. Then the improper integral

$$\int_1^\infty \frac{M(x)}{x^{s+1}} \, dx$$

converges uniformly on every compact subset of the half-plane $\sigma > \Theta$, and it therefore defines an analytic function on that domain. By analytic continuation, it coincides with $(s\zeta(s))^{-1}$. This means that $\zeta(s)$ has no zeros in $\sigma > \Theta$.