MA3150 ANALYTIC NUMBER THEORY, EXERCISES 2, 2019: THE GAMMA FUNCTION AND THE FUNCTIONAL EQUATION FOR $\zeta(s)$

(1) Show by explicit computation that $\Gamma(1/2) = \sqrt{\pi}$.

(2) Show that
$$\int_1^\infty (x - [x])x^{-2}\,dx = 1 - \gamma,$$
where γ is Euler's constant.

(3) Use the result of problem (2) to show that
$$\lim_{s\to 1}\left(\frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1}\right) = \gamma.$$
Hint: Remember that $\zeta(s)$ can be expressed as
$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty (x - [x])x^{-s-1}\,dx.$$

(4) Use the infinite product representation

(1) $$1/\Gamma(s) = se^{\gamma s}\prod_{k=1}^{\infty}(1 + s/k)e^{-s/k}$$

and the result of problem (3) to show that
$$\lim_{s\to 0}\left(\frac{\zeta'(1-s)}{\zeta(1-s)} + \frac{\Gamma'(s)}{\Gamma(s)}\right) = 0.$$

(5) Use the functional equation $\Gamma(s+1) = s\Gamma(s)$, the reflection formula
$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s},$$
and Legendre's duplication formula
$$\Gamma(2s) = \frac{2^{2s-1}}{\sqrt{\pi}}\Gamma(s)\Gamma(s+1/2),$$
to establish the unsymmetric form of the functional equation:
$$\zeta(1-s) = 2^{1-s}\pi^{-s}(\cos\frac{1}{2}\pi s)\Gamma(s)\zeta(s).$$

(6) Use the unsymmetric form of the functional equation to compute the logarithmic derivative of $\zeta(1-s)$. Use this formula and the result of problem (4) to show that $\zeta'(0)/\zeta(0) = \log(2\pi)$.

(7) Determine the residues of $x^s\zeta'(s)/(s\zeta(s))$ at all its real poles.

(8) Let C be the rectangular contour (oriented counter-clockwise) with corners at the points $c \pm iT$ and $-(2N+1) \pm iT$, where $c > 1$ and N is a positive integer. Assume that $\zeta(s)$ has no zeros on the horizontal line $t = T$. Use the solution to problem (7) to express
$$\frac{1}{2\pi i}\int_C \frac{x^s\zeta'(s)}{s\zeta(s)}\,ds$$

in terms of the nontrivial zeros ρ_j of $\zeta(s)$ with imaginary parts between $-T$ and T.

(9) Use Stirling's formula to prove that

$$\frac{\Gamma'(s)}{\Gamma(s)} = \log s + O(1)$$

when s is at positive distance from the poles of $\Gamma(s)$. Can you describe a region in which we have the sharper asymptotic formula

$$\frac{\Gamma'(s)}{\Gamma(s)} = \log s + O\left(|s|^{-1}\right)?$$

(10) Use the logarithmic derivative from problem (6) and the result of problem (9) to show that

$$|\zeta'(s)/\zeta(s)| = O(\log(|s|+2)$$

when s at distance say 1 from the trivial zeros of $\zeta(s)$ and $\sigma \le -1$.

Remark. If you succeeded in solving the problems above, you did a good part of the work required to establish von Mangoldt's explicit formula for Chebyshev's function $\psi^*(x)$. What remains in our analysis of the above contour integral, is to deal with that part of the contour C that comes close to the nontrivial zeros of $\zeta(s)$.