THE ZEROS OF $\zeta(s)$

Peter Lindqvist 2008

It follows from the Euler product

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - p^{-s}\right)^{-1}$$

that $\zeta(s) \neq 0$ in the half-plane $\sigma = \mathrm{Re}(s) > 1$. A more sophisticated proof yields that $\zeta(1+it) \neq 0$, when $-\infty < t < \infty$. (But in each strip $1 < \sigma < 1+\delta$ the function $\zeta(s)$ attains every complex value except 0 infinitely often!) The functional equation

$$\boxed{\pi^{-\frac{s}{2}}\,\Gamma\!\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\,\Gamma\!\left(\frac{1-s}{2}\right)\zeta(1-s)}$$

shows that, except the trivial zeros at $-2, -4, -6, -8, \ldots$, all zeros are located in the critical strip $0 < \sigma < 1$. The notation

$$\rho_k = \beta_k + i\gamma_k$$

is customary for the non-trivial zeros. By symmetry ρ_k, $1-\rho_k$, $\overline{\rho_k}$, $1-\overline{\rho_k}$ are zeros. There are no zeros on the segment $[0,1]$ of the real axis; this follows from

$$\zeta(\sigma) = \left(1 - \frac{1}{2^{\sigma-1}}\right)\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^\sigma}$$

and $\zeta(\sigma) < 0$, when $0 < \sigma < 1$. The first zero is

$$\frac{1}{2} + i \cdot 14{,}134725 \qquad \text{(Gram 1903)}$$

and it lies on the critical line with the abscissa ½. By now it has been numerically verified that the first 100 billion zeros are situated on the critical line.

THE RIEMANN HYPOTHESIS The zeros of $\zeta(s)$ in the strip $0 < \sigma < 1$ are all on the line $\mathrm{Re}(s) = \frac{1}{2}$.

So far as I know, the Riemann Hypothesis has never been proved. Notice that on the critical line

$$\rho = \frac{1}{2} + i\gamma, \quad 1 - \rho = \bar{\rho}.$$

The explicit formula

$$\psi(x) = x - \sum \frac{x^{\rho_k}}{\rho_k} + \frac{1}{2}\log\left(\frac{x^2}{x^2-1}\right) - \log(2\pi)$$

for Tschebyschev's summatory function

$$\psi(x) = \frac{1}{2}\left(\sum_{n<x} \Lambda(n) + \sum_{n \leq x} \Lambda(n)\right)$$

illustrates the rôle of the hypothesis, since pairing the roots $\frac{1}{2} \pm i\gamma$ yields

$$\frac{x^{\rho}}{\rho} + \frac{x^{\bar{\rho}}}{\bar{\rho}} = x^{\frac{1}{2}} \frac{\cos(\gamma \log x) + 2\gamma \sin(\gamma \log x)}{\frac{1}{4} + \gamma^2} .$$

Then one could obtain at least that

$$\psi(x) = x + O_\varepsilon(x^{\frac{1}{2}+\varepsilon})$$

where $\varepsilon > 0$ is as small as we please.

In order to represent $\zeta(s)$ as a product with factors containing the zeros we begin with the entire function

$$\boxed{\xi(s) = \frac{s(s-1)}{2} \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s),}$$

which is easier to handle. It has the functional equation

$$\underline{\xi(s) = \xi(1-s).}$$

Recall that

$$\frac{1}{\Gamma(\frac{s}{2})} = \frac{s}{2} e^{\frac{\gamma s}{2}} \prod_{n=1}^{\infty} \left(1 + \frac{s}{2n}\right) e^{-\frac{s}{2n}} .$$

Thus the factor $\Gamma(\frac{s}{2})$ in the formula for $\xi(s)$ cancells the trivial zeros $s = -2n$. Therefore the zeros of $\xi(s)$ are exactly the non-trivial zeros of $\zeta(s)$.

Recall Jensen's formula

$$\log|f(0)| + \sum_{|z_k|<R} \log\frac{R}{|z_k|} = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\,d\theta$$

for an entire function $f(z)$ with zeros at $z_1, z_2, \ldots$ ($f(0) \neq 0$ above).

LEMMA For $M(r) = \max_{|s|=r} |\xi(s)|$ we have

$$\log(M(r)) \sim \frac{1}{2} r \log(r)$$

as $r \to \infty$. In other words, the order of growth is 1.

Proof: First, when $s = r > 2$ we get the lower estimate

$$M(r) \geq \frac{r(r-1)}{2} \pi^{-\frac{r}{2}} \Gamma\left(\frac{r}{2}\right) \zeta(r)$$

$$\geq \pi^{-\frac{r}{2}} \Gamma\left(\frac{r}{2}\right)$$

and Stirling's formula $\log\Gamma\left(\frac{r}{2}\right) \sim \frac{r}{2}\log\frac{r}{2}$ shows that

$$\underline{\lim}_{r\to\infty} \frac{\log M(r)}{\frac{r}{2}\log(r)} \geq 1.$$

Second, we prove that

$$\log|\xi(s)| \leq \frac{|s|}{2}\log|s| + A|s|$$

when $\sigma \geq \frac{1}{2}$ and $|s| \geq 2$. Then it easily follows from $\xi(s) = \xi(s-1)$ that the above inequality also holds when $\sigma \leq \frac{1}{2}$, $|s| \geq 2$, but with another constant A. Then one gets

$$\overline{\lim_{r \to \infty}}\left(\frac{\log M(r)}{\frac{r}{2}\log(r)}\right) \leq 1.$$

To this end, keep $\sigma \geq \frac{1}{2}$, and consider

$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \frac{\{x\}}{x^{s+1}}\,dx.$$

Now

$$|\zeta(s)| \leq \frac{|s|}{|s|-1} + |s|\int_1^\infty \frac{dx}{x^{1+\sigma}}$$

$$\leq |s| + \frac{|s|}{\sigma} \leq 3|s|$$

The major contribution comes from

$$\log\Gamma\left(\tfrac{s}{2}\right) \sim \frac{s}{2}\log\frac{s}{2} \qquad \text{(Stirling)}$$

$|\arg(\frac{s}{2})| < \frac{\pi}{2}$ $$\log\left|\Gamma\left(\tfrac{s}{2}\right)\right| \sim \frac{\sigma}{2}\log\frac{|s|}{2} - \frac{t}{2}\arg\left(\frac{s}{2}\right)$$

$$\lesssim \frac{|s|}{2}\log|s| + a|s|$$

and an easy calculation concludes the proof. □

Now we apply Jensen's formula in the disc $|s| \leq 2R$. Since $\xi(0) = \xi(1) = \frac{1}{2}$,

$$\log \frac{1}{2} + \sum_{|\rho_k| < 2R} \log \frac{2R}{|\rho_k|} \lesssim \frac{1}{2} \cdot 2R \log(2R)$$

It follows that the number of roots in the disc $|s| \leq R$ satisfies

$$\underline{n(R) \leq C R \log R}$$

when R is large. (For these roots $\log \frac{2R}{|\rho_k|} \geq \log(2)$.)

COR. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{|\rho_k|^\alpha} < \infty$, if $\alpha > 1$

<u>Proof</u>: The sum is

$$\sum_{j=1}^{\infty} \sum_{2^{j-1} \leq |\rho_k| < 2^j} |\rho_k|^{-\alpha} \leq \sum_{j=1}^{\infty} \left(\frac{1}{2^{j-1}}\right)^\alpha C \cdot 2^j \log(2^j)$$

and the majorant converges. ∎

A similar technique leads to

$$\boxed{\sum_{|\rho_k| < \lambda} \frac{1}{|\rho_k|} = O(\log^2 \lambda), \quad \alpha = 1 \qquad \sum_{|\rho_k| \geq \lambda} \frac{1}{|\rho_k|^2} = O\left(\frac{\log \lambda}{\lambda}\right), \quad \alpha = 2}$$

both needed later. (Start with $\lambda = 2^N$.)

According to Hadamard's theorem

$$\boxed{\xi(s) = e^{a+bs} \prod_{k=1}^{\infty} \left(1 - \frac{s}{\rho_k}\right) e^{+\frac{s}{\rho_k}}}$$

where the product is absolutely convergent. It converges uniformly in bounded domains.

COR $\sum \frac{1}{|\rho_k|} = \infty$. In particular, there are infinitely many roots ρ_k.

Proof: If the sum were convergent, $\sigma >> 1$,

$$|\xi(\sigma)| \leq e^{|a|+|b|\sigma} \prod \underbrace{\left|1 - \frac{\sigma}{\rho_k}\right|}_{\leq 1 + \frac{\sigma}{|\rho_k|} \leq e^{\sigma/|\rho_k|}} e^{\frac{\sigma}{|\rho_k|}}$$

$$\leq e^{|a|+|b|\sigma} e^{2\sigma \sum \frac{1}{|\rho_k|}} = e^{A + B\sigma}$$

This contradicts $\log|\xi(\sigma)| \sim \frac{\sigma}{2} \log(\sigma)$ as $\sigma \longrightarrow \infty$ ∎

The constants are

$$e^{a} = \frac{1}{2} = \xi(0)$$

Euler's

$$b = \frac{1}{2}\log(4\pi) - 1 - \frac{\gamma}{2} \approx -0{,}023\ldots$$

As an exercise we determine b by logarithmic differentiation:

$$\frac{\xi'(s)}{\xi(s)} = b + \sum_k \left\{ \frac{1}{s-\rho_k} + \frac{1}{\rho_k} \right\}$$

$\xi(s) = \xi(1-s)$

$$\frac{\xi'(0)}{\xi(0)} = b. \qquad \text{But } \frac{\xi'(0)}{\xi(0)} = -\frac{\xi'(1)}{\xi(1)}.$$

$$\frac{\xi'(s)}{\xi(s)} = \frac{1}{s} + \frac{1}{s-1} - \frac{1}{2}\log\pi + \frac{\Gamma'(\frac{s}{2})}{2\Gamma(\frac{s}{2})} + \frac{\zeta'(s)}{\zeta(s)},$$

$$\frac{\xi'(1)}{\xi(1)} = 1 + \left[\frac{\frac{d}{ds}(s-1)\zeta(s)}{(s-1)\zeta(s)}\right]_{s=1} - \frac{1}{2}\log\pi + \frac{\Gamma'(\frac{1}{2})}{2\Gamma(\frac{1}{2})}.$$

$$-\frac{d}{ds}\log\Gamma\left(\frac{s}{2}\right) = \frac{1}{s} + \frac{\gamma}{2} + \sum_{n=1}^{\infty}\left\{\frac{1}{s+2n} - \frac{1}{2n}\right\}$$

$$-\frac{\Gamma'(1/2)}{2\Gamma(1/2)} = 1 + \underset{\text{Euler's } \gamma}{\frac{\gamma}{2}} + \underbrace{\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5} - \frac{1}{4} + \cdots\right)}_{\log(2) - 1}$$

$$\zeta(s) = \frac{1}{s-1} + \gamma + c_1(s-1) + c_2(s-1)^2 + \cdots$$

$$(s-1)\zeta(s) = 1 + \gamma(s-1) + c_1(s-1)^2 + \cdots$$

$$\frac{\frac{d}{ds}(s-1)\zeta(s)}{(s-1)\zeta(s)} = \frac{0+\gamma+2c_1(s-1)+\cdots}{1+\gamma(s-1)+\cdots} \longrightarrow \gamma$$

as $s \to 1$.

In toto

$$\frac{\xi'(1)}{\xi(1)} = 1+\gamma-\frac{1}{2}\log\pi-\frac{\gamma}{2}-\log 2$$

$$= 1+\frac{\gamma}{2}-\frac{1}{2}\log(4\pi),$$

which is $-b$.

An interesting consequence is

$$\frac{\xi'(1)}{\xi(1)} = b + \sum\left\{\frac{1}{1-\rho_k}+\frac{1}{\rho_k}\right\}$$

$$= b + 2\sum_k \frac{1}{\rho_k}$$

Conditionally convergent sum!

since $1-\rho_k$ is a root whenever ρ_k is a root.

$$\boxed{\lim_{\substack{T\to\infty}}\sum_{|\gamma_k|<T}\frac{1}{\rho_k} = 1+\frac{\gamma}{2}-\frac{\log(4\pi)}{2} = 0{,}02309\ldots}$$

Riemann wrote down 20 decimal places!

The order of summation is important, since $\sum|\rho_k|^{-1}=\infty$.

Combining the formulas

$$\frac{s}{2}\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\left[(s-1)\zeta(s)\right] = \frac{1}{2}e^{bs}\prod\left(1-\frac{s}{\rho_k}\right)e^{\frac{s}{\rho_k}},$$

$$\frac{1}{\Gamma\left(\frac{s}{2}\right)} = \frac{s}{2}e^{\frac{\gamma s}{2}}\prod\left(1+\frac{s}{2n}\right)e^{-\frac{s}{2n}},$$

$$b = \frac{1}{2}\log(4\pi) - 1 - \frac{\gamma}{2}$$

we arrive at

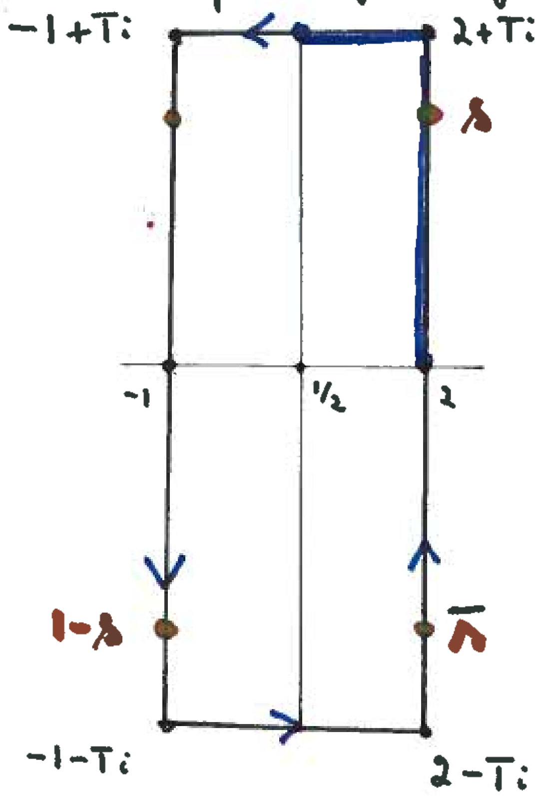
$$\boxed{(s-1)\zeta(s) = \frac{1}{2}\left(\frac{2\pi}{e}\right)^{s}\prod_{n=1}^{\infty}\left(1+\frac{s}{2n}\right)e^{-\frac{s}{2n}} \times \prod_{k=1}^{\infty}\left(1-\frac{s}{\rho_k}\right)e^{+\frac{s}{\rho_k}}}$$

The first product exhibits the trivial zeros $s=-2n$, the second is over the roots in the critical strip.

How many zeros are there? Let $N(T)$ denote the total number of zeros (counted with multiplicity) in the rectangle $0 \le \sigma \le 1$, $0 \le t \le T$ above the real axis. It was proved by von Mangoldt that

$$N(T) = \frac{T}{2\pi}\log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi} + O(\log T)$$

The formula was given without error term by Riemann. The proof is based on the Principle of Argument.



$$2N(T) = \frac{1}{2\pi i}\oint \frac{\xi'(s)}{\xi(s)}\,ds$$

$$= \frac{4}{2\pi i}\int_L \frac{\xi'(s)}{\xi(s)}\,ds$$

where L goes from 2 to $2+Ti$ to $-1+Ti$. This comes from the symmetries

$$\xi(s) = \xi(1-s),$$

$$\xi(\bar{s}) = \overline{\xi(s)}.$$

The hardest part is the integral of $\frac{\zeta'(s)}{\zeta(s)}$. We leave this aside.

When the zeros above the real axis are arranged according to $\gamma_{n+1} \geq \gamma_n$, then

$$|\rho_n| \sim \gamma_n \sim \frac{2\pi n}{\log(n)},$$

where $\rho_n = \beta_n + i\gamma_n$.

Hardy and Littlewood proved in 1914 that there are infinitely many zeros on the critical line. They considered the function

$$Z(t) = \zeta\left(\tfrac{1}{2}+it\right) \frac{\Gamma\left(\frac{1}{4}+i\frac{t}{2}\right)\pi^{-\frac{1}{4}-\frac{it}{2}}}{\left|\Gamma\left(\frac{1}{4}+i\frac{t}{2}\right)\pi^{-1/4-\frac{it}{2}}\right|}$$

This function takes only real values! Moreover its zeros are related to the zeros $\frac{1}{2}+i\gamma$ on the critical line. Also

$$|Z(t)| = \left|\zeta\left(\tfrac{1}{2}+it\right)\right|.$$

Notice that $Z(t)$ changes its sign whenever we have a zero $\frac{1}{2}+i\gamma$ of odd order. If

$$\left|\int_T^{2T} Z(t)\,dt\right| \underset{\text{STRICT}}{<} \int_T^{2T} |Z(t)|\,dt$$

then $Z(t)$ must change its sign in the interval $T < t < 2T$, which detects the presence of a zero. H.-L. proved the above inequality for $T \longrightarrow \infty$.

Selberg proved that

$$N_0(T) \geq c\,T \log T \qquad (T \to \infty)$$

where $N_0(T)$ is the number of the zeros on the critical line, $\rho = \frac{1}{2} + i\gamma$, $0 < \gamma \leq T$. Later

$$N_0(T) \geq \frac{1}{3} N(T) \qquad \text{Levinson 1974}$$

$$N_0(T) \geq \frac{2}{5} N(T) \qquad \text{Conrey 1989}$$

Recall the formula

$$\int_0^x \psi(u)\,du = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(-\frac{\zeta'(s)}{\zeta(s)}\right) \frac{x^{s+1}}{s(s+1)}\,ds$$

where $c > 1$. If the product formula with the roots is logarithmically differentiated and $\zeta'(s)/\zeta(s)$ is inserted a longer calculation yields

$$\boxed{\int_0^x \psi(t)\,dt = \frac{x^2}{2} - \sum_{k=1}^{\infty} \frac{x^{\rho_k+1}}{\rho_k(\rho_k+1)} - \sum_{n=1}^{\infty} \frac{x^{1-2n}}{2n(2n-1)} - \frac{\zeta'(0)}{\zeta(0)}\,x + \frac{\zeta'(-1)}{\zeta(-1)}}$$

This is an "exact formula". The calculation is based on the differentiated formula

$$-\frac{\zeta'(s)}{\zeta(s)} = \frac{s}{s-1} - \sum_{\rho} \frac{s}{\rho(s-\rho)} + \sum_{n=1}^{\infty} \frac{s}{2n(s+2n)} - \frac{\zeta'(0)}{\zeta(0)}$$

from which we read off

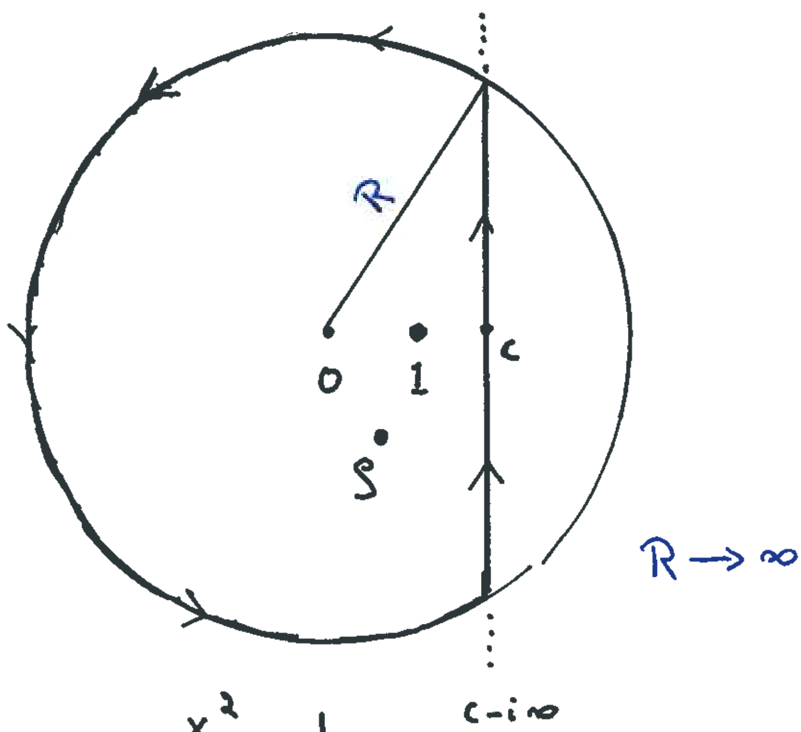
$$\sum_{\rho} \frac{1}{\rho(1+\rho)} = \frac{\zeta'(-1)}{\zeta(-1)} + \frac{1}{2} - \sum_{n=1}^{\infty} \frac{1}{2n(2n-1)} - \frac{\zeta'(0)}{\zeta(0)}$$

for later use.

We split the integral

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(-\frac{\zeta'(s)}{\zeta(s)}\,\frac{x^{s+1}}{s(s+1)}\right)ds$$

in three*) parts.



$$\frac{1}{2\pi i}\oint \frac{\cancel{s}}{s-1}\,\frac{x^{s+1}}{\cancel{s}(s+1)}\,ds = \frac{x^2}{2} - \frac{1}{2}$$

Residues at $s=1$, $s=0$

$$\frac{1}{2\pi i}\oint \frac{-1\cdot x^{s+1}}{\rho(s-\rho)(s+1)}\,ds = \underset{s=-1}{\frac{1}{\rho(\rho+1)}} - \underset{s=\rho}{\frac{x^{\rho+1}}{\rho(\rho+1)}}$$

if $R > |\rho|$.

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*) Also the $-\zeta'(0)/\zeta(0)$ term must be taken into account. Thus we have four integrals.

$$\frac{1}{2\pi i}\oint \frac{x^{s+1}}{2n(s+2n)}\frac{1}{s+1}\,ds = \underset{s=-1}{\frac{1}{2n(2n-1)}} - \underset{s=-2n}{\frac{x^{1-2n}}{2n(2n-1)}}$$

if $R > 2n$.

The integrals over the arcs (to the left) approach 0 as $R \to \infty$. Integrating term-wise*) we find

$$\int_0^x \psi(u)\,du = \frac{x^2}{2} - \frac{1}{2} + \sum_\rho \frac{1}{\rho(\rho+1)} - \sum_\rho \frac{x^{\rho+1}}{\rho(\rho+1)}$$
$$+ \sum_{n=1}^\infty \frac{1}{2n(2n-1)} - \sum_{n=1}^\infty \frac{x^{1-2n}}{2n(2n-1)} - \frac{\zeta'(0)}{\zeta(0)}(x-1)$$

$$= \frac{x^2}{2} - \sum_\rho \frac{x^{\rho+1}}{\rho(\rho+1)} + \frac{\zeta'(-1)}{\zeta(-1)}$$
$$- \sum_{n=1}^\infty \frac{x^{1-2n}}{2n(2n-1)} - \frac{\zeta'(0)}{\zeta(0)}x$$

where the intermediate formula (with $s=-1$) was used. This is the exact formula.

Next we differentiate the formula with respect to x and obtain

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*) The sums converge absolutely and uniformly in $t \in \mathbb{R}$.

$$\boxed{\psi(x) = x - \sum \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\log\left(1-\frac{1}{x^2}\right)}$$

after summing $\sum_{n=1}^{\infty} \frac{x^{-2n}}{2n}$. The justification of

$$\frac{d}{dx}\left(\sum \frac{x^{\rho+1}}{\rho(\rho+1)}\right) = \sum \frac{x^\rho}{\rho}$$

is not clear. (But the formula is CONDITIONALLY CONVERGENT true.)

<u>Remark</u> $\frac{\zeta'(0)}{\zeta(0)} = \log(2\pi)$.

Since the sum

$$\sum_\rho \frac{x^\rho}{\rho} = \lim_{L\to\infty} \sum_{|\rho|<L} \frac{x^\rho}{\rho}$$

is difficult to handle (for example, it has discontinuities at all points x = prime powers !!!), we proceed via the averaged Tschebyschew function $\psi_1(x) = \int_1^x \psi(u)\,du$, where $x > 1$.

Let us from now on make the

HYPOTHESIS*) $\Theta = \sup_\rho (\operatorname{Re}\rho) < 1$.

In any case $\frac{1}{2} \leq \Theta \leq 1$ and we only know

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*) Such a hypothesis has, so far as I know, never been proved. We know that $\Theta \leq 1$ and that there are no zeros on the line $\sigma = 1$.

that the closed half-plane $\sigma \geqq 1$ is free of zeros. The Riemann Hypothesis is that $\Theta = \frac{1}{2}$.

Under the Hypothesis

$$\sum_{\rho} \left| \frac{x^{\rho+1}}{\rho(\rho+1)} \right| = \sum_{\rho} \frac{x^{1+\beta}}{|\rho||\rho+1|} \leq x^{1+\Theta} \sum_{\rho} \frac{1}{|\rho||\rho+1|}$$

where the last sum converges, because $\sum |\rho|^{-2}$ converges. It follows that

$$\psi_1(x) = \frac{x^2}{2} + \mathcal{O}(x^{1+\Theta})$$

THEOREM*) If $\zeta(s) \neq 0$, when the abscissa $\sigma > \Theta$, then

$$\begin{cases} \psi(x) = x + \mathcal{O}(x^{\Theta} \log^2 x), \\ \vartheta(x) = x + \mathcal{O}(x^{\Theta} \log^2 x), \\ \pi(x) = \operatorname{li}(x) + \mathcal{O}(x^{\Theta} \log x). \end{cases}$$

Under the Riemann Hypothesis

$$\pi(x) = \operatorname{li}(x) + \mathcal{O}(\sqrt{x} \log x).$$

von Koch

Proof: The passage from $\psi_1(x)$ to $\psi(x)$ is the delicate part. Avoiding the demanding differentiation we write the difference

*) The theorem is worthless for $\Theta = 1$.

$$\psi_1(x+1) - \psi_1(x) = \int_x^{x+1} \psi(u)\,du = \psi(x) + O(\log x)$$

$$\psi_1(x+1) - \psi_1(x) = x + \frac{1}{2} - \sum_\rho \frac{(x+1)^{\rho+1} - x^{\rho+1}}{\rho(\rho+1)} - \frac{\zeta'(0)}{\zeta(0)} + O\left(\frac{1}{x}\right)$$

in two ways. The first comes from "$\log p \le \log(x+1)$" and the second follows from the explicit formula. We estimate the sum in two ways, depending on the size of $|\rho|$. Denote

$$\omega_\rho = \frac{(x+1)^{\rho+1} - x^{\rho+1}}{\rho(\rho+1)} .$$

Then

$$|\omega_\rho| = \left|\frac{1}{\rho}\int_x^{x+1} u^\rho\,du\right| \le 2\frac{x^\Theta}{\rho} \qquad (\text{use for } |\rho| \le \lambda)$$

$$|\omega_\rho| \le \frac{(x+1)^{\Theta+1} + x^{\Theta+1}}{|\rho(\rho+1)|} \le C\frac{x^{1+\Theta}}{|\rho|^2} \qquad (\text{use for } |\rho| \ge \lambda)$$

Thus the sum is

$$\sum_\rho |\omega_\rho| = \sum_{|\rho|\le\lambda} |\omega_\rho| + \sum_{|\rho|>\lambda} |\omega_\rho| \le 2x^\Theta \sum_{|\rho|<\lambda} \frac{1}{|\rho|}$$

$$+ C x^{1+\Theta} \sum_{|\rho|\ge\lambda} \frac{1}{|\rho|^2} \quad \le$$

$$\leq x^{\theta} O(\log^2 \lambda) + x^{1+\theta} O\left(\frac{\log \lambda}{\lambda}\right)$$

according to the estimates for truncated sums on page 6. Taking $\log \lambda = x$ we obtain

$$\sum_{\rho} \left| \frac{(x+1)^{\rho+1} - x^{\rho+1}}{\rho(\rho+1)} \right| \leq C_1 x^{\theta} \log^2 x .$$

The estimate $\psi(x) = x + O(x^{\theta} \log^2 x)$ follows.

Since

$$\vartheta(x) \leq \psi(x) \leq \vartheta(x) + O(\sqrt{x} \log x)$$

always holds (Tschebyschev), we have also established that

$$\vartheta(x) = x + O(x^{\theta} \log^2 x) .$$

The passage to $\pi(x)$ goes via the formula[†)]

$$\pi(x) = \frac{\vartheta(x)}{\log(x)} + \int_2^x \frac{\vartheta(u)}{u \log^2 u} .$$

Writing $\vartheta(x) = x + \varepsilon(x)$ we obtain

$$\pi(x) = \int_2^x \frac{du}{\log u} + \frac{2}{\log 2} + \underbrace{\int_2^x \frac{\varepsilon(u)}{u \log^2 u} du}_{O(x^{\theta})}$$

$$+ \underbrace{\frac{\varepsilon(x)}{\log(x)}}_{O(x^{\theta} \log x)}$$

This concludes the proof. □

†) See Apostol.

Under the Riemann Hypothesis we have

$$\pi(x) = li(x) + O(\sqrt{x}\log x).$$ • von Koch 1903

On the other hand, it is known that the estimate

$$\pi(x) = li(x) + O\left(\frac{\sqrt{x}}{\log x}\right)$$

is false. — If the error is of the magnitude $O_\varepsilon(x^{\frac{1}{2}+\varepsilon})$, for each $\varepsilon > 0$, then the Riemann Hypothesis is true.[†)]

The exact formula implies the Prime Numbers Theorem, because

$$\lim_{x\to\infty}\left(\sum_{\rho}\frac{x^{\rho+1}}{\rho(\rho+1)} : \frac{x^2}{2}\right) = 0.$$

This follows from considering

$$\sum_{\rho} = \sum_{|\rho|\le\lambda} + \sum_{|\rho|>\lambda}.$$

Considering zero-free regions like $\sigma \geq 1 - \frac{a}{\log(t)}$ one obtains better estimates[*)], like

$$\pi(x) = li(x) + O\left(x e^{-c\sqrt{\log x}}\right)$$ • de la Vallée-Poussin 1896

Without any hypothesis one has even[**)]

$$\pi(x) = li(x) + O\left(x e^{-(\log x)^{3/5}/(\log\log x)^{1/5}}\right).$$

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*) The error is of smaller magnitude than any estimate like $O\left(\frac{x}{(\log x)^K}\right)$ where $K >> 1$ is arbitrary.

†) See Nevanlinna – P., Complex Analysis.

**) $\frac{3}{5} > \frac{1}{2}$ (in Γ)