ESTIMATE OF $\zeta(s)$

$$\boxed{\zeta(s) = 1 + \frac{1}{s-1} - s\int_1^x \underbrace{(y-[y])}_{\{y\}} y^{-1-s}\,dy}$$

"fractional part"

VALID FOR $\sigma > 0$.

This comes from the Euler-Maclaurin summability formula for $\sum n^{-s}$. We obtain

$$\zeta(s) = \sum_{n\le x}\frac{1}{n^s} + \frac{x^{1-s}}{s-1} + \{x\}x^{-s} - s\int_x^\infty \{y\}y^{-s-1}\,dy,$$

from which the following estimates come (choose $x = t$!)

LEMMA: $\boxed{|\zeta(s)| \le A\log(t)}$, $\boxed{|\zeta'(s)| \le A(\log(t))^2}$, $\sigma \ge 1$ and $t \ge 2$

$\boxed{|\zeta(s)| \le A_\delta t^{1-\delta}}$ $(\sigma \ge \delta > 0,\ t \ge 1)$

LEMMA: The residue of $\zeta'(s)/\zeta(s)$ at $s = 1$ is

$$\boxed{\lim_{s\to 1}(s-1)\frac{\zeta'(s)}{\zeta(s)} = -1}$$

RESIDUE

Proof: $\zeta(s) = 1 + \frac{1}{s-1} - s\int_1^x \{y\}y^{-1-s}\,dy$

$\zeta'(s) = -\frac{1}{(s-1)^2} - \int_1^x \{y\}y^{-1-s}\,dy + s\int_1^x \{y\}\{y\}^{-1-s}\log(y)\,dy$

The integrals are finite when $s = 1$. The limit follows. $\square$

ZERO FREE REGION $\sigma \geq 1$.

PROPOSITION $\zeta(s) \neq 0$ when $\sigma > 1$.

Proof $\left|\frac{1}{\zeta(s)}\right| = \prod_p \left|1 - \frac{1}{p^s}\right| \leq \prod_p \left(1 + \frac{1}{p^\sigma}\right) < \zeta(\sigma)$

$\Rightarrow |\zeta(s)| > \frac{1}{\zeta(\sigma)} > 0$ when $\sigma > 1$.

[One can also use $\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$ (conv. when $\sigma > 1$)]

THM $\zeta(1+it) \neq 0$, $-\infty < t < \infty$ $(t \neq 0)$.

Proof. $\sigma > 1$

$$\log \zeta(s) = \sum -\log(1-p^{-s}) = \sum_{m,p} \frac{p^{-ms}}{m} = \sum_{n=2}^{\infty} c_n n^{-s}$$

where $c_n \geq 0$. (In fact, $c_n = \frac{\Lambda(n)}{\log(n)}$)

- $\log \zeta(s) = \log|\zeta(s)| + i \arg(\zeta(s))$, since $\zeta(s) \neq 0$ in the half-plane $\sigma > 1$.

Hence

$$\log|\zeta(s)| = \operatorname{Re} \sum_{n=2}^{\infty} \cdots = \sum_{n=2}^{\infty} c_n \underbrace{n^{-\sigma} \cos(t \log n)}_{\operatorname{Re}(n^{-s})}$$

$$\log|\zeta(\sigma)^3 \zeta(\sigma+it)^4 \zeta(\sigma+2it)| = 3\log|\zeta(\sigma)| + 4\log|\zeta(\sigma+it)| +$$

$$+ \log|\zeta(\sigma+2it)| = \sum c_n n^{-\sigma} \underbrace{(3 + 4\cos(t\log n) + \cos(2t\log n))}_{\geq 0}$$

Since $2(1+\cos(\theta))^2 = 3 + 4\cos\theta + \cos(2\theta)$.

But $\log|\ldots| \geq 0 \Rightarrow$

$$\boxed{|\zeta(\sigma)^3 \zeta(\sigma+it)^4 \zeta(\sigma+2it)| \geq 1, \quad \sigma > 1}$$

We need to $\sigma \to 1+$. Then $\frac{1}{\sigma-1} < \zeta(\sigma) < \frac{\sigma}{\sigma-1}$ by integral test.

To proceed, assume that

ANTITHESIS $\zeta(1+it_0) = 0$ for some $t_0 \neq 0$.

$$\underbrace{\frac{1}{\sigma-1}}_{\to\infty} \le \underbrace{|(\sigma-1)\zeta(\sigma)|^3}_{\substack{\le \sigma^3 \\ \text{BOUNDED TERM}}} \underbrace{\left|\frac{\zeta(\sigma+it_0) - \overbrace{\zeta(1+it_0)}^{=0}}{\sigma-1}\right|^4}_{\to \zeta'(1+it_0)} \underbrace{|\zeta(\sigma+2it_0)|}_{\to \zeta(1+2it_0)}$$

This is a contradiction, since the right-hand side remains bounded as $\sigma \to 1+$. $\lightning$. $\square$

We can extract an estimate from the proof:

LEMMA $\left|\frac{1}{\zeta(s)}\right| \le A(\log t)^7$, $\sigma \ge 1$, $t \ge t_0$.

<u>Proof</u>: The case $\sigma \ge 2$ is clear. Take first $\sigma \neq 1$, $1 < \sigma < 2$.

$(t \ge 2)$

$$(\sigma-1)^3 \le \underbrace{|(\sigma-1)\zeta(\sigma)|}_{\le \sigma^3 \le 8} \, |\zeta(s)|^4 \underbrace{|\zeta(\sigma+2it)|}_{\le A_1 \log(2t) \le 2A_1 \log(t)}$$

$$|\zeta(\sigma+it)| \ge A_3 \frac{(\sigma-1)^{3/4}}{\log(t)^{1/4}}$$

Select an auxiliary η, $1 \le \sigma < \eta \le 2$ (now $\sigma = 1$ is included). Then

$$\zeta(\eta+it) - \zeta(\sigma+it) = \int_\sigma^\eta \zeta'(u+it)\,du$$

$$|\zeta(\eta+it) - \zeta(\sigma+it)| \le A_4 (\log t)^2 (\eta-\sigma)$$
$$\le A_4 (\log t)^2 (\eta-1)$$

$$|\zeta(\sigma+it)| \ge |\zeta(\eta+it)| - A_4(\eta-1)(\log t)^2$$
$$\ge A_3 \frac{(\eta-1)^{3/4}}{(\log t)^{1/4}} - A_4(\eta-1)(\log t)^2$$

The green estimate is valid also if $1 < \eta < \sigma \le 2$ since now

$$|\zeta(\sigma+it)| \ge A_3 \frac{(\sigma-1)^{3/4}}{(\log t)^{1/4}} \ge A_3 \frac{(\eta-1)^{3/4}}{(\log t)^{1/4}}.$$

Now, choose $\eta - 1$ in

$$|\zeta(\sigma+it)| \ge A_3 \frac{(\eta-1)^{3/4}}{(\log t)^{1/4}} - A_4(\eta-1)(\log t)^2$$

so that

$$A_3 \frac{(\eta-1)^{3/4}}{(\log t)^{1/4}} = 2A_4(\eta-1)(\log t)^2$$

$$\eta = 1 + \left(\frac{A_3}{2A_4}\right)^4 \frac{1}{(\log t)^9}$$

Now $1 < \eta \le 2$, provided that $t \ge t_0$ (a number). This value of η yields

$$|\zeta(\sigma+it)| \ge \frac{A_5}{(\log t)^7} \quad (\sigma \ge 1,\ t \ge t_0)$$

$\square$

COROLLARY $\boxed{\left|\frac{\zeta'(s)}{\zeta(s)}\right| \le A(\log t)^9}$, $(\sigma \ge 1, t \ge t_0)$

Proof: We also had $|\zeta'(s)| \le A'(\log t)^2$, $\sigma \ge 1$, $t \ge 2$. $\square$

TRANSITION FROM $\pi(x)$ TO A FORMULA WITH $\zeta'(s)/\zeta(s)$.

We know

$$\pi(x) \sim \frac{x}{\log(x)} \iff \underset{\text{Tchebyshef}}{\theta(x) \sim x} \iff \underset{\text{Tchebyshef}}{\psi(x) \sim x}$$

However, $\psi(x) = \sum_{n \le x} \Lambda(n)$ is a discontinuous step function. It is more convenient to use $\int_2^x \psi(u)\,du$, which is continuous.

LEMMA

$$\psi_1(x) \equiv \int_2^x \psi(u)\,du \sim \frac{x^2}{2} \iff \psi(x) \sim x.$$

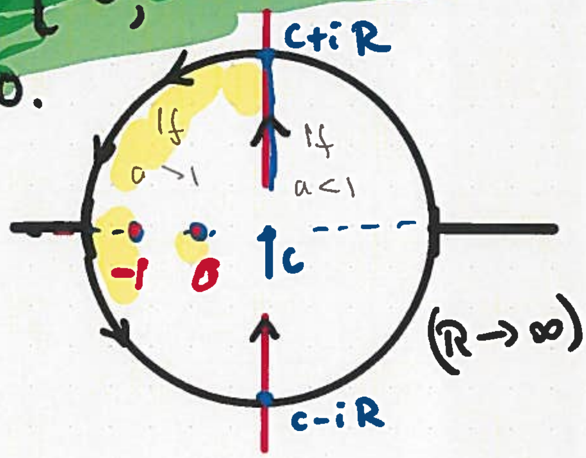
Proof: The simple proof uses only the fact that $\psi(x)$ is monotone. [See T. Apostol §13.$^{4}_{2}$].

LEMMA

Later $a = \frac{x}{n}$. $z = s$ [§13.2]

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{a^z}{z(z+1)}\,dz = \begin{cases} 1-\frac{1}{a}, & a \geq 1 \\ 0, & 0 < a \leq 1 \end{cases}$$

when $c > 0$.



Proof. Use the half circle contour, the left one (center $= c$) if $a > 1$. The residues at $z = 0$, $z = -1$ count. ∎

[§13.2 Lemma 1]

$$\psi_1(x) = \int_2^x \psi(u)\,du = \sum_{n \leq x} (x-n)\Lambda(n) \qquad \Big(\psi(x) = \sum_{n\leq x} \Lambda(n),$$

use Abel summability formula for $\sum n\Lambda(n)$)

$$= x\sum_{n \leq x}\left(1 - \frac{n}{x}\right)\Lambda(n) = \frac{1}{2\pi i}\, x \sum_{n=2}^{\infty} \Lambda(n) \int_{c-i\infty}^{c+i\infty} \frac{\left(\frac{x}{n}\right)^s ds}{s(s+1)}$$

The integral is zero when $n > x$. (A finite number of terms!)

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\overbrace{\left(\sum_{n=2}^{\infty}\frac{\Lambda(n)}{n^s}\right)}^{= -\frac{\zeta'}{\zeta}\,!}\, x^{s+1}}{s(s+1)}\,ds$$

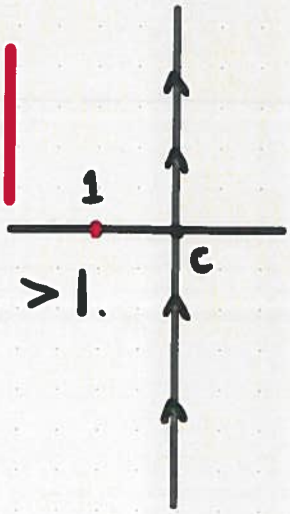
where $c > 1$ so that $\sum\int = \int\sum$. (ABSOL. conv.)

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right) ds$$

THEOREM

$$\psi_1(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right) ds$$

where $c > 1$.

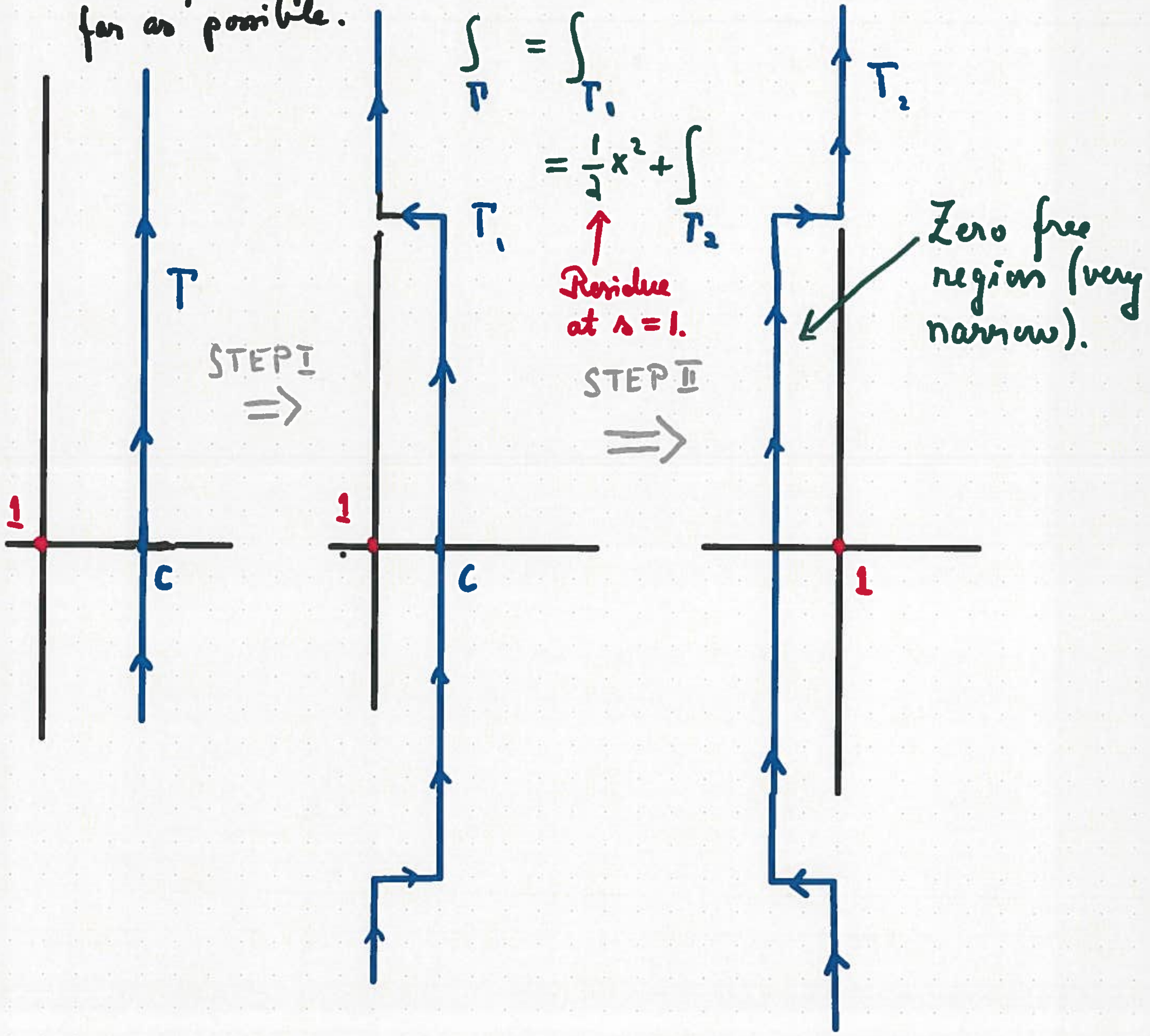


<u>Difficulty</u>: $|x^{s+1}| = x^{2+(c-1)} >> x^2$, $c > 1$.

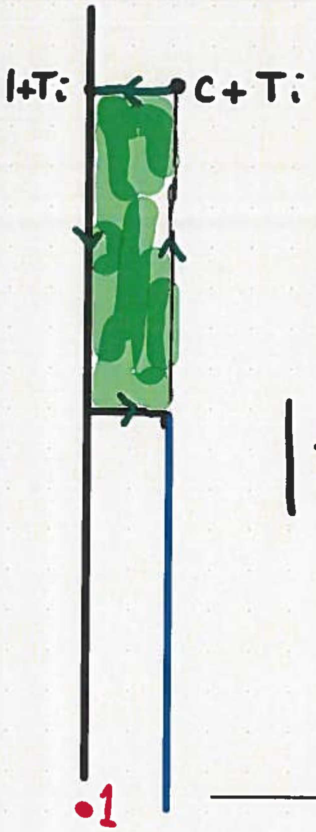
but we want $\psi_1(x) \sim \frac{1}{2}x^2$. Thus the line of integration must be moved to the left as far as possible.

$$\int_{T} = \int_{T_1} = \frac{1}{2}x^2 + \int_{T_2}$$

Residue at $s = 1$.



STEP I



The integral $\oint$ along the rectangle is $= 0$ (Cauchy) and we have only to verify that

$$\int_{C+Ti}^{1+Ti} \dots ds \longrightarrow 0 \text{ as } T \to +\infty$$

Indeed

$$\left| \frac{1}{2\pi i} \int_{C+Ti}^{1+Ti} \frac{x^{s+1}}{s(s+1)} \left(-\frac{\zeta'(s)}{\zeta(s)} \right) d\sigma \right| \leq \frac{x^{C+1}}{2\pi\, T^2} A(\log t)^9 \xrightarrow[T \to \infty]{} 0$$

$s = \sigma + Ti$, $ds = d\sigma$

$$|s(s+1)|^2 \geq (\sigma^2 + T^2)((c+1)^2 + T^2) \geq T^4$$

STEP II

Zero free rectangle.

The integral along the rectangle is

$$\frac{1}{2\pi i} \oint \dots ds = \operatorname*{Res}_{s=1} \left\{ \frac{x^{s+1}}{s(s+1)} \left(-\frac{\zeta'(s)}{\zeta(s)} \right) \right\}$$

$$= \frac{x^{1+1}}{1(1+1)} \underbrace{\lim_{s \to 1} (s-1) \left[-\frac{\zeta'(s)}{\zeta(s)} \right]}_{=1}$$

$$= \frac{1}{2} x^2$$

Now we have achieved:

$$\boxed{\psi_1(x) = \frac{1}{2} x^2 + \frac{1}{2\pi i} \int_{\zeta} \frac{x^{s+1}}{s(s+1)} \left(-\frac{\zeta'(s)}{\zeta(s)} \right) ds}$$

ZEROS The zeros of an analytic function are ISOLATED. A bounded domain can contain only a FINITE number of zeros. Since $\zeta(1+it) \neq 0$, we can make the rectangle so narrow that the zeros are outside it!