ANALYTIC NUMBER THEORY

(1) Given n, construct n consecutive composite numbers

$$N+1,\ N+2,\ \dots,\ N+n.$$

(2) We define

$$li(x) = \lim_{\varepsilon \to 0+} \left\{ \int_0^{1-\varepsilon} \frac{dt}{\log(t)} + \int_{1+\varepsilon}^{2} \frac{dt}{\log(t)} \right\} + \int_2^x \frac{dt}{\log(t)}$$

$$= \int_2^x \frac{dt}{\log(t)} - 1{,}04\dots$$

Prove that the principal value of the integral exists. (Prove that the limit exists).

(3) Show that

$$|\zeta(s)| \le \zeta(\sigma),\quad s = \sigma + it,\quad \sigma > 1.$$

(4) Verify the formula

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} - s\int_N^\infty \frac{x-[x]}{x^{s+1}}\,dx + \frac{N^{1-s}}{s-1}$$

when $\sigma = \mathrm{Re}(s) > 1$. (Actually, the formula is valid when $\sigma > 0$.)

(5) $$\sum_{n=1}^{\infty} \frac{\mu(n)^2}{n^2} = \frac{\zeta(2)}{\zeta(4)} \quad \left(= \frac{15}{\pi^2} \right)$$

"square-free numbers"