SOLUTIONS

① The numbers

$n!+2,\ n!+3,\ n!+4,\ \ldots,\ n!+n \qquad (n \geq 2)$

are all composite numbers. For example, $3 \mid n!+3$.

② $$\int_0^{1-\varepsilon} \frac{dx}{\ln(x)} = \int_\varepsilon^1 \frac{dy}{\ln(1-y)}$$ (the singularity at $x=0$ or $y=1$ is ignorable)

$$\int_{1+\varepsilon}^2 \frac{dx}{\ln(x)} = \int_\varepsilon^1 \frac{dy}{\ln(1+y)}$$

$li(2) = 1{,}0451637801\ldots$

$$\int_{\substack{0 \\ |t-1| \geq \varepsilon}}^2 \frac{dt}{\ln(t)} = \int_\varepsilon^1 \frac{\ln(1-y^2)}{\ln(1+y)\cdot\ln(1-y)}\,dy$$

The integrand has a finite limit as $y \to 0+$:

$$\frac{\ln(1-y^2)}{\ln(1+y)\cdot\ln(1-y)} = \frac{-y^2 - \frac{y^4}{2} - \cdots}{\left(y - \frac{y^2}{2} + \cdots\right)\left(-y - \frac{y^2}{2}\cdots\right)}$$

$$= \frac{1 + y^2/2 + \cdots}{1 - y^2/4 + \cdots} \longrightarrow 1 \quad \text{as } y \to 0+$$

It follows that the limit exists as $\varepsilon \to 0+$.

③ $n^{-s} = e^{-(\sigma+it)\operatorname{Log}(n)}$, $|n^{-s}| = e^{-\sigma \operatorname{Log}(n)} = n^{-\sigma}$

$$|\zeta(s)| \leq \sum_{n=1}^{\infty} |n^{-s}| = \sum_{n=1}^{\infty} n^{-\sigma} = \zeta(\sigma), \quad \sigma > 1$$

④ Euler-Maclaurin for $\sum_{n=1}^{N} n^{-s}$. (Apostol, Thm 12.21 with $a=0$. Page 269)

p. 56

(5)

$$\frac{\zeta(2)}{\zeta(4)} = \frac{\prod_p \left(1-\frac{1}{p^4}\right)}{\prod_p \left(1-\frac{1}{p^2}\right)} = \prod_p \left(1+\frac{1}{p^2}\right)$$

$$= 1 + \sum \frac{1}{p_k^2} + \sum \frac{1}{p_k^2 p_j^2} + \cdots$$

$$= \sum_{n \text{ is "square free"}} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{|\mu(n)|^2}{n^2}$$

---

§1/16

$$2^{nm} - 1 = (2^m)^n - 1$$

$$= (2^m - 1)(2^{m(n-1)} + 2^{m(n-2)} + \cdots + 2^m + 1)$$

This is a factorization of $2^{nm}-1$ unless $m=1$ or $n=1$.

---

§1/17

$$\frac{2^{kn}+1}{2^n+1} = 2^{(k-1)n} - 2^{(k-2)n} + \cdots + 1 \quad \text{if } k = \text{odd}.$$

---

§1/30 Harmonic numbers $H_k = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}$.

Only H_1 is an integer. If $k = p$ (a prime number), then

$$p!\,H_p = 2\cdot 3\cdot 4\cdots p + 1\cdot 3\cdot 4\cdots p + 1\cdot 2\cdot 4\cdots p + \cdots + 1\cdot 2\cdot 3\cdots (p-1)$$

exhibits that H_p cannot be an integer. Every term has the factor p except the very last one.

The general case is less transparent. But using BERTRAND'S POSTULATE we choose a prime number p, $\frac{k}{2} < p < k$. Again all terms in

$$k!\,H_k = 2\cdot 3\cdot 4\cdots k + \cdots + 1\cdot 2\cdots (p-1)(p+1)\cdots k + \cdots + 1\cdot 2\cdots p\cdots (k-1)$$

can be divided by p except one of them. ($2p$ comes too late).

3.1

$$\sum_{n \le x} \frac{\log(n)}{n} = \int_1^x \frac{\log(t)}{t}\,dt + \int_1^x \frac{1-\log(t)}{t^2}\{t\}\,dt - \underbrace{(x-[x])}_{\{x\}} \frac{\log(x)}{x}$$

$$f(x) = \frac{\log(x)}{x}$$

$$= \frac{1}{2}(\log(x))^2 + \underbrace{\int_1^\infty \frac{1-\log(t)}{t^2}\{t\}\,dt}_{= A \text{ (a constant)}} - \underbrace{\int_x^\infty \frac{1-\log(t)}{t^2}\{t\}\,dt}_{\text{Abs value } \le}$$

Euler-Maclaurin

$$\int_x^\infty \frac{\log(t)}{t} = \frac{\log x}{x}$$

$0 \le \{t\} < 1$

$$+ O\left(\frac{\log x}{x}\right)$$

$$= \frac{1}{2}(\log(x))^2 + A + O\left(\frac{\log x}{x}\right)$$

3.1

$$\sum_{n=2}^{x} \frac{1}{n\log(n)} = \int_2^x \frac{dt}{t\log(t)} - (x-[x])\frac{1}{x\log x} + \frac{1}{2\log(2)}$$

$$- \int_2^x \frac{1+\log(t)}{t^2(\log(t))^2}\{t\}\,dt$$

$x \ge 3$

$$= \log(\log x) + O\left(\frac{1}{x\log(x)}\right) - \underbrace{\int_2^\infty \frac{1+\log(t)}{t^2(\log(t))^2}\{t\}\,dt}_{\text{A constant, say } -B.}$$

$$+ \underbrace{\int_x^\infty \frac{1+\log(t)}{t^2(\log t)^2}\{t\}\,dt}_{O\left(\frac{1}{x\log(x)}\right)}$$