(16) Establish the formula

$$\Theta(x) = \sum_{n=1}^{\infty} \mu(n)\, \psi(x^{\frac{1}{n}})$$

Hint: Möbius inversion.

(17) Construct the Dirichlet characters $\chi_1, \chi_2, \ldots, \chi_6$ modulo 9.

(18) Verify

$$\prod_{p}\left(1 + \frac{1}{p(p+1)}\right) = \frac{315}{2\pi^4}\zeta(3) = 1{,}943596\ldots$$

(19) Prove that

$$-\frac{1}{1-\sigma} < \zeta(\sigma) < -\frac{\sigma}{1-\sigma} \qquad (0<\sigma<1).$$

Thus $\zeta(\sigma) < 0$, when $0<\sigma<1$.

(20) Show that

$$\zeta(s) = -s\int_0^{\infty} \frac{x-[x]}{x^{s+1}} \qquad (0 < \mathrm{Re}(s) < 1)$$

(21) Show that

$$\sum_p \frac{1}{p^s} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log(\zeta(ns)), \quad s > 1.$$

(22) Gauss' circle problem. Show that the number of lattice points (m,n), $m,n = 0, \pm 1, \pm 2, \dots$ in the disc $m^2 + n^2 \le R$ is

$$= \pi R + O(\sqrt{R}) \quad \text{as } R \to \infty.$$

(23) Suppose that $f(n)$ is a multiplicative function. The formula

$$\sum_{n=1}^{\infty} f(n) = \prod_p (1 + f(p) + f(p^2) + \cdots)$$

is valid provided that $\sum_{n=1}^{\infty} |f(n)| < \infty$. (Theorem 11.6, Apostol p. 230). Verify that the formula is false for

$$f(p) = -1, \quad f(p^2) = 1 + p^{-2}, \quad f(p^\nu) = 0 \text{ if } \nu \ge 3$$

although the corresponding product converges absolutely.

(24) Off side. Which is larger:

$$\int_0^1 x^x \, dx \quad \text{or} \quad \int_0^1 \int_0^1 (xy)^{xy} \, dx \, dy \quad ?$$