Ex 11/3 $\quad \sum_{n=1}^{N} n^{-1-it} \quad (t \neq 0)$

$$\sum_{n=1}^{N} f(n) = f(1) + \int_1^N f(x)\,dx + \int_1^N (x-[x])\, f'(x)\,dx$$

$$f(x) = x^{-1-it}, \quad f'(x) = -(1+it)\, x^{-2-it}$$

$$\sum_{n=1}^{N} \frac{1}{n^{1+it}} = 1 + \underbrace{\left.\frac{x^{-it}}{-it}\right|_1^N}_{\frac{i}{t}(N^{-it}-1)} + \int_1^N \frac{(1+it)\{x\}}{x^{2+it}}\,dx$$

a) $\left| \sum_{n=1}^{N} \frac{1}{n^{1+it}} \right|$

$$\leq 1 + \frac{1}{|t|}\cdot 2 + (1+|t|)\int_1^N \frac{dx}{x^2} = 1 + \frac{2}{|t|} + (1+|t|)\left(1-\frac{1}{N}\right)$$

$$\leq 2 + \frac{2}{|t|} + |t|$$

Hence the partial sums are bounded, uniformly in N.

b) The last integral is even absolutely convergent as $N \to \infty$. Indeed,

$$\left| \int_1^N \frac{\{x\}\,dx}{x^{2+it}} \right| \leq \int_1^N \frac{dx}{x^2} \leq \int_1^\infty \frac{dx}{x^2} = 1.$$

$\lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n^{1+it}}$ exists $\iff \lim_{N\to\infty} (N^{-it})$ exists.

It does not!

$$N^{-it} = e^{-it\log(N)}$$

CONTRADICTION TO CAUCHY'S CRITERION.

$$|N^{-it} - (2N)^{-it}| = |e^{-it\log(N)} - e^{-it\log(2N)}|$$

$$= |e^{-it\log(N)}(1 - e^{-it\log 2})|$$

$$= |1 - e^{-it\log(2)}| \neq 0$$ provided that $t\log(2) \neq 2k\pi$. Then $\lim (N^{-it})$ does not exist. If $t\log(2) = 2k\pi$, then use $(3N)^{-it}$ in place of $(2N)^{-it}$.

<u>CONCLUSION</u> $\sum_{n=1}^{\infty} n^{-1-it}$ is divergent.