(18)

$$1+\frac{1}{p(p-1)} = 1+\frac{1}{p^2(1-\frac{1}{p})} = \frac{1-\frac{1}{p}+\frac{1}{p^2}}{1-\frac{1}{p}}$$

$$= \frac{(1-\frac{1}{p}+\frac{1}{p^2})}{(1-\frac{1}{p})}\cdot\frac{(1+\frac{1}{p})}{1+\frac{1}{p}} = \frac{1+\frac{1}{p^3}}{1-\frac{1}{p^2}}$$

$$= \frac{(1+\frac{1}{p^3})(1-\frac{1}{p^3})}{(1-\frac{1}{p^2})(1-\frac{1}{p^3})} = \frac{1-1/p^6}{(1-\frac{1}{p^2})(1-\frac{1}{p^3})}$$

Hence

$$\prod_p\left(1+\frac{1}{p(p-1)}\right) = \frac{\prod_p(1-\frac{1}{p^6})}{\prod_p(1-\frac{1}{p^2})\cdot\prod_p(1-\frac{1}{p^3})}$$

$$= \frac{\zeta(2)\zeta(3)}{\zeta(6)} = \frac{\frac{\pi^2}{6}}{\frac{\pi^6}{945}}\zeta(3) = \frac{315}{2\pi^4}\zeta(3)$$

since

$$\zeta(s) = \frac{1}{\prod_p(1-\frac{1}{p^s})}, \quad s>1.$$

<u>Remark</u> $$\prod_p\left(1+\frac{1}{p(p+1)}\right) = \frac{\zeta(2)}{\zeta(3)}$$

(+)

(21) $$\sum_p \frac{1}{p^s} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log \zeta(ns), \qquad s > 1$$

Preparation $$\sum_{n=1}^{\infty} \frac{\mu(n)\, x^n}{1-x^n} = x \qquad (0 < x < 1)$$ LAMBERT SERIES

since $$\sum_{n=1}^{\infty} \frac{\mu(n)\, x^n}{1-x^n} = \sum_{n=1}^{\infty} \mu(n)(x^n + x^{2n} + x^{3n} + \dots)$$

$$= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \mu(n)\, x^{mn} = \sum_{N=1}^{\infty} x^N \underbrace{\sum_{d \mid N} \mu(d)}_{\substack{=0 \text{ when} \\ N \geq 2}} = x^1 \cdot 1 = x.$$

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{t^{ns}-1} = \frac{1}{t^s}, \quad t > 1 \qquad (\text{write } x = t^{-s}) \quad \square$$

Use $$\frac{1}{s} \log \zeta(s) = \int_2^{\infty} \frac{\pi(t)\, dt}{t\,(t^s-1)}$$

$$\frac{1}{ns} \log \zeta(ns) = \int_2^{\infty} \frac{\pi(t)\, dt}{t\,(t^{ns}-1)}$$

and sum: ${}^{\dagger)}$

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log \zeta(ns) = s \int_2^{\infty} \sum \frac{\mu(n)}{t^{ns}-1} \frac{\pi(t)}{t}\, dt$$

†) absolute conv.

$$= s \int_2^{\infty} \frac{\pi(t)\, dt}{t^{s+1}} = \sum_p \frac{1}{p^s}$$ (Abel sum)

(24) The integrals are equal.