Exercise 1, MA3150 Spring 2019

(b),(c) p. 7

A.2.1 (a) $\varphi(n) = \frac{n}{2} \iff \frac{1}{2} = \prod_{p|n}\left(1 - \frac{1}{p}\right)$

$\iff 2 \mid n$. Hence $n = p^k$, $k \geq 1$

A.2.2 (a) $(m, n) = 1 \implies (\varphi(m), \varphi(n)) = 1$

because if $p \mid \varphi(m)$ then $p \mid m$.

(b) n composite $\not\Rightarrow$ $(n, \varphi(n)) > 1$.

Counter-example: $\varphi(15) = 8$.

(c) Suppose $p \mid m \iff p \mid n$. Then

$$n\,\varphi(m) = n \cdot m \prod_{p|m}\left(1 - \frac{1}{p}\right) = n \cdot m \prod_{p|n}\left(1 - \frac{1}{p}\right)$$

$$= m\,\varphi(n).$$

A.2.6 We write $n = \prod_{j=1}^{l} p_j^{\alpha_j}$.

Then $d^2 \mid n \iff d \mid \prod_{j=1}^{l} p_j^{[\frac{\alpha_j}{2}]}$.

The latter product is 1 $\iff$ n is square-free. Hence

$$\sum_{d^2|n} \mu(d) = |\mu(n)|.$$

More generally, $d^k \mid n$

$\iff d \mid \prod_{j=1}^{l} p_j^{[\alpha_k/k]}$, where the product is 1 $\iff$ no m^k divides n. Hence

$$\sum_{d^k | n} \mu(d) = \begin{cases} 1, & m^k | n \text{ for some } m^k \\ 0, & \text{otherwise.} \end{cases}$$

A.2.21. We set $f(n) := [\sqrt{n}\,] - [\sqrt{n-1}\,]$.

We observe that $f(n) = \begin{cases} 1, & n = m^2 \\ 0, & \text{otherwise.} \end{cases}$

If $(m, n) = 1$, then mn is a square $\iff$ both m and n are squares.

Hence f is multiplicative. But

$$1 = f(4) \neq f(2) f(2) = 0,$$ so

f is not completely multiplicative.

A.2.26. In view of Thm. 2.17, it suffices to check that $f^{-1}(p) = \mu(p) f(p)$ $= -f(p)$. But this is clear because

$$f^{-1}(p) \cdot f(1) + f^{-1}(1) f(p) = 0$$

and $f(1) = f^{-1}(1) = 1$.

A.3.1 (a) $\sum_{n \le x} \frac{\log n}{n} = \sum_{1 < x \le n} \frac{\log n}{n}$

$$= \int_1^x \frac{\log y}{y}\, dy + \int_1^x (y - [y]) \cdot \left\{ \frac{\log y - 1}{y^2} \right\} dy + O\left(\frac{\log x}{x}\right)$$

$$= \frac{1}{2}(\log x)^2 + A + O\left(\frac{\log x}{x}\right) + O\left(\int_x^\infty \frac{\log y}{y^2}\, dy\right)$$

$$= \frac{1}{2}(\log x)^2 + A + O\left(\frac{\log x}{x}\right),$$ where

$$A := \int_1^\infty (y - [y]) \left\{ \frac{\log y - 1}{y^2} \right\} dy.$$

(b) $\sum_{2 \le n \le x} \frac{1}{n \log n} = \frac{1}{2 \log 2} + \sum_{2 < n \le x} \frac{1}{n \log n}$

$$= \frac{1}{2\log 2} + \int_2^x \frac{dy}{y \log y} - \int_2^x (y - [y]) \cdot \frac{\log y + 1}{(y \log y)^2}\, dy$$

$$+ O\left(\frac{1}{x \log x}\right) = \log\log x + A + O\left(\frac{1}{x \log x}\right),$$

where $A := \frac{1}{2\log 2} - \log\log 2 - \int_2^\infty (y - [y]) \frac{\log y + 1}{(y \log y)^2} dy.$

A.3.2. We have that

$$\sum_{n \le x} d(n) = x \log x + (2\gamma - 1)x + O(\sqrt{x}).$$

Hence by Abel summation,

$$\sum_{n \le x} \frac{d(n)}{n} = \log x + O(1)$$

$$+ \int_1^x \left(y \log y + (2\gamma - 1)y + O(\sqrt{y})\right) \cdot \frac{dy}{y^2}$$

$$= \frac{1}{2}(\log x)^2 + 2\gamma \log x + O(1).$$

A.3.4 (a) $\left[\frac{x}{n}\right]^2 = \frac{x^2}{n^2} + O\left(\frac{x}{n}\right), \quad x \geq n.$

Hence $\sum_{n \leq x} \mu(n) \left[\frac{x}{n}\right]^2 = x^2 \sum_{n \leq x} \frac{\mu(n)}{n^2}$

$+ O\left(x \sum_{n \leq x} \frac{1}{n}\right) = \frac{x^2}{\zeta(2)} + O(x \log x).$

(b) Similarly, we get

$\sum_{n \leq x} \frac{\mu(n)}{n} \left[\frac{x}{n}\right] = x \sum_{n \leq x} \frac{\mu(n)}{n^2} + O(\log x)$

$= \frac{x}{\zeta(2)} + O(\log x).$

A.3.5 (a) We may compute as before that $\sum_{n \leq x} \varphi(n) = \sum_{d \leq x} \mu(d) \sum_{q \leq \frac{x}{d}} q.$

Now $\sum_{q \leq \frac{x}{d}} q = \sum_{q \leq [\frac{x}{d}]} q = \frac{1}{2}\left[\frac{x}{d}\right] \cdot \left(\left[\frac{x}{d}\right] + 1\right)$

So $\sum_{n \leq x} \varphi(n) = \frac{1}{2} \sum_{n \leq x} \mu(n) \left[\frac{x}{d}\right]^2$

$+ \frac{1}{2} \sum_{n \leq x} \mu(n) \left[\frac{x}{d}\right] \underset{\text{Thm 3.12}}{=} \frac{1}{2} \sum_{n \leq x} \mu(n) \left[\frac{x}{d}\right]^2 + \frac{1}{2}.$

(b) By Möbius inversion, $\frac{\varphi(n)}{n} = \sum_{d \mid n} \frac{\mu(d)}{d}.$

Hence

$\sum_{n \leq x} \frac{\varphi(n)}{n} = \sum_{n \leq x} \sum_{d \mid n} \frac{\mu(d)}{d} = \sum_{d \leq x} \frac{\mu(d)}{d} \sum_{q < \frac{x}{d}} 1$

$= \sum_{d \leq x} \frac{\mu(d)}{d} \left[\frac{x}{d}\right].$

A.4.17 We have $a_1 < a_2 < \dots < a_n \le x$ and $a_i \nmid \prod_{\substack{j=1 \\ j\neq i}}^{n} a_j$. This implies that there exist n distinct primes $p_1, \dots, p_n$ with the property that $p_i \mid a_i$ but $p_i \nmid a_j$, $j \neq i$. Hence we would reach a contradiction if $n > \pi(x)$.

A.4.18 (a) By Abel summation,

$$\pi(x) = \sum_{p \le x} \frac{\log p}{\log p} = \frac{\theta(x)}{\log x} - \int_2^x \frac{\theta(y)}{(\log y)^2 y}\, dy.$$

If $\theta(x) = x + E(x)$, then since

$$\int_2^x \frac{dx}{\log x} = \frac{x}{\log x} - \frac{2}{\log 2} - \int^x \frac{dy}{(\log y)^2}$$

by integration by parts, we get

$$\pi(x) = \int_2^x \frac{dx}{\log x} + \frac{E(x)}{\log x} + O\left(\int_2^x \frac{|E(y)|}{(\log y)^2 y}\, dy\right).$$

This is the right way of expressing the relation between $\pi(x)$ and $\theta(x)$!

(b) A similar calculation as above.

A 4.19 (a) We found above that

$$Li(x) = \frac{x}{\log x} + \int_2^x \frac{dt}{(\log t)^2} - \frac{2}{\log 2},$$

so the formula holds for $n = 1$.

By integration by parts,

$$\int_2^x \frac{dt}{(\log t)^{n+1}} = \frac{x}{(\log x)^{n+1}} - \frac{2}{(\log 2)^{n+1}}$$

$$+ (n+2) \int_2^x \frac{dt}{(\log t)^{n+2}}$$

and hence the formula holds for all $n \geq 1$ by induction, with

$$C_n = -2 \sum_{j=1}^{n} \frac{(j-1)!}{(\log 2)^j}.$$

(b) $\varphi(n) = \varphi(2n) \iff \prod_{p|n} \left(1 - \frac{1}{p}\right) = 2 \prod_{p|2n} \left(1 - \frac{1}{p}\right)$

$\iff 2 \nmid n$.

(c) $\varphi(n) = 12 \iff 12 = \prod_{p|n} \left(p^{\alpha_p} - p^{\alpha_p - 1}\right)$,

where $n = \prod_{p|n} p^{\alpha_p}$.

We consider the divisors of 12:

$1 = 2\left(1 - \frac{1}{2}\right)$

$12 = 13\left(1 - \frac{1}{13}\right)$

$4 = 5\left(1 - \frac{1}{5}\right) = 8 \cdot \left(1 - \frac{1}{2}\right)$

$6 = 7 \cdot \left(1 - \frac{1}{7}\right) = 9 \cdot \left(1 - \frac{1}{3}\right)$

$3 \neq p^{\alpha_p} - p^{\alpha_p - 1}$ for all primes p, $\alpha_p \geq 1$.

$2 = 3\left(1 - \frac{1}{3}\right) = 4\left(1 - \frac{1}{2}\right)$. Hence:

13, 26, 21, 42, 28, 36.