Exercise. Prove the inequality

$$\Big(\int_{\mathbb{R}^d} |x|^2 |f(x)|^2 \, dx\Big) \cdot \Big(\int_{\mathbb{R}^d} |\xi|^2 |\hat{f}(\xi)|^2 \, d\xi\Big) \geq \frac{d^2}{16\pi^2} \|f\|^4_{L^2(\mathbb{R}^d)} \qquad (*)$$

where $f \in L^2(\mathbb{R}^d)$.

Show that the lower bound is attained if and only if $f(x) = A \exp(-B|x|^2)$, $A \in \mathbb{C}$, $B > 0$.

[Hint: Show $(*)$ first for f a Schwartz function on $\mathbb{R}^d$. Approximate a general $f \in L^2(\mathbb{R}^d)$ by a sequence of Schwartz functions. See Ch. 5.4 (p. 158–159) of Stein & Shakarchi: *Fourier Analysis*, for the case $d = 1$. (Copies handed out.)]