

The Riesz-Markov-Kakutani theorem

MA3105 Advanced Real Analysis

Norwegian University of Science and Technology (NTNU)

Formulation of the theorem in the simplest case

Denote by $C[0, 1]$ the vector space of **continuous** functions $f: [0, 1] \rightarrow \mathbb{R}$. With the uniform norm, $C[0, 1]$ is a Banach space.

Let $I: C[0, 1] \rightarrow \mathbb{R}$ be a positive linear functional. There exists a unique positive, finite Borel measure μ on $[0, 1]$ such that

$$I(f) = \int_{[0,1]} f d\mu$$

for all $f \in C[0, 1]$.

The idea of the proof

Given $I: C[0, 1] \rightarrow \mathbb{R}$ a positive linear functional, want a measure μ so that for all $f \in C[0, 1]$

$$I(f) = \int_{[0,1]} f d\mu.$$

So in some sense, it is like we had the integral and wanted to find the measure.

Then for Borel sets E , we would like to define $\mu(E)$ as

$$\mu(E) = \int_{[0,1]} \mathbb{1}_E d\mu := I(\mathbb{1}_E).$$

The problem is that $\mathbb{1}_E$ is **not** continuous in general, so $I(\mathbb{1}_E)$ is not defined.

The idea is then to **extend** I to a larger space of functions, that contains at least all indicator functions $\mathbb{1}_U$, where U is open.

The extension of I to a larger space

We denote by $\text{BLSC}_+[0, 1]$ the family of all **bounded, lower semi continuous** functions $f: [0, 1] \rightarrow \mathbb{R}$ with $f \geq 0$.

It is easy to see that for every open set $U \subset [0, 1]$, the function $\mathbb{1}_U \in \text{BLSC}_+[0, 1]$.

Functions in $\text{BLSC}_+[0, 1]$ are well approximated from below by continuous functions. Then it makes sense to define, for every $f \in \text{BLSC}_+[0, 1]$,

$$\tilde{I}(f) := \sup\{I(g) : g \in C[0, 1] \text{ and } 0 \leq g \leq f\}.$$

The properties of the extension

For every $f \in \text{BLSC}_+[0, 1]$,

$$\tilde{I}(f) := \sup\{I(g) : g \in C[0, 1] \text{ and } 0 \leq g \leq f\}.$$

- (i) If $f \in C[0, 1]$ then $\tilde{I}(f) = I(f)$ (\tilde{I} indeed extends I).
- (ii) For all $f \in \text{BLSC}_+[0, 1]$, $\tilde{I}(f) \geq 0$ (positivity).
- (iii) If $f_1 \leq f_2$ then $\tilde{I}(f_1) \leq \tilde{I}(f_2)$ (monotonicity).
- (iv) For all $f \in \text{BLSC}_+[0, 1]$, $\tilde{I}(f) < \infty$ (finiteness).
- (v) For all $c \geq 0$, $\tilde{I}(cf) = c\tilde{I}(f)$ (homogeneity).
- (vi) For all $f_1, f_2 \in \text{BLSC}_+[0, 1]$, $\tilde{I}(f_1 + f_2) \geq \tilde{I}(f_1) + \tilde{I}(f_2)$ (super-additivity).
- (vii) If f and $f_1, f_2, \dots \in \text{BLSC}_+[0, 1]$ are such that $f \leq \sum_{n=1}^{\infty} f_n$ then $\tilde{I}(f) \leq \sum_{n=1}^{\infty} \tilde{I}(f_n)$ (countable sub-additivity).

Towards defining the measure

For every **open** set $U \subset [0, 1]$, $\mathbb{1}_U \in \text{BLSC}_+[0, 1]$, so we may define

$$\mu_0(U) := \tilde{I}(\mathbb{1}_U).$$

Note: In fact we can show (homework) that

$\mu_0(U) = \sup\{I(g) : g \in C[0, 1], 0 \leq g \leq 1 \text{ and } \text{supp}(g) \subset U\}$,
where $\text{supp}(g)$ is the **topological support** of g .

It is easy to verify that μ_0 satisfies the following:

- $\mu_0(\emptyset) = 0$ and $0 \leq \mu_0(U) < \infty$ for every open set U .
- If $U_1 \subset U_2$ then $\mu_0(U_1) \leq \mu_0(U_2)$.
- If $U = \bigcup_{n=1}^{\infty} U_n$ then $\mu_0(U) \leq \sum_{n=1}^{\infty} \mu_0(U_n)$.

The last property follows from the countable sub-additivity of the extension \tilde{I} .

Defining the outer measure

For every **open** set $U \subset [0, 1]$ we have defined

$$\mu_0(U) := \tilde{I}(\mathbb{1}_U).$$

Now for every set $E \subset [0, 1]$ define

$$\mu^*(E) := \inf\{\mu_0(U) : U \text{ open, } U \supset E\}.$$

Note that if U is an open set, then $\mu^*(U) = \mu_0(U)$.

It is not difficult to show that μ^* is an **outer-measure**, meaning that it satisfies the following properties:

- (1) $\mu^* \geq 0$ and $\mu^*(\emptyset) = 0$.
- (2) If $E_1 \subset E_2$ then $\mu^*(E_1) \leq \mu^*(E_2)$.
- (3) If $E = \bigcup_{n=1}^{\infty} E_n$ then $\mu^*(E) \leq \sum_{n=1}^{\infty} \mu^*(E_n)$.

At last, the promised measure

By Carathéodory's extension theorem, given the outer measure μ^* , the family of sets $A \subset [0, 1]$ with the property

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A) \text{ for all } E \subset [0, 1]$$

forms a σ -algebra.

Moreover, restricted to this σ -algebra, the outer measure μ^* is a measure, which we shall denote by μ .

It turns out that **all open sets** verify the above property. Then μ is a Borel measure, and it is finite.

Finally, we have to check that this measure does the job, that is, for all $f \in C[0, 1]$ we have $I(f) = \int_{[0,1]} f d\mu$.

This follows from the fact that continuous functions can be approximated by simple functions of the form $\sum_{i=1}^k c_i \mathbb{1}_{E_i}$ with E_1, \dots, E_k **open or closed** sets.