The Riesz-Markov-Kakutani theorem

MA3105 Advanced Real Analysis

Norwegian University of Science and Technology (NTNU)

Formulation of the theorem in the simplest case

Denote by $C[0,1]$ the vector space of continuous functions $f\colon [0,1] \to \mathbb{R}$. With the uniform norm, $C[0,1]$ is a Banach space.

Let $I\colon C[0,1] \to \mathbb{R}$ be a positive linear functional. There exists a unique positive, finite Borel measure μ on $[0,1]$ such that

$$I(f) = \int_{[0,1]} f\, d\mu$$

for all $f \in C[0,1]$.

The idea of the proof

Given $I\colon C[0,1] \to \mathbb{R}$ a positive linear functional, want a measure μ so that for all $f \in C[0,1]$

$$I(f) = \int_{[0,1]} f \, d\mu.$$

So in some sense, it is like we had the integral and wanted to find the measure.

Then for Borel sets E, we would like to define $\mu(E)$ as

$$\mu(E) = \int_{[0,1]} \mathbb{1}_E \, d\mu := I(\mathbb{1}_E).$$

The problem is that $\mathbb{1}_E$ is not continuous in general, so $I(\mathbb{1}_E)$ is not defined.
The idea is then to **extend** I to a larger space of functions, that contains at least all indicator functions $\mathbb{1}_U$, where U is open.

The extension of I to a larger space

We denote by $\mathrm{BLSC}_+[0,1]$ the family of all bounded, lower semi continuous functions $f\colon [0,1] \to \mathbb{R}$ with $f \geqslant 0$.

It is easy to see that for every open set $U \subset [0,1]$, the function $\mathbb{1}_U \in \mathrm{BLSC}_+[0,1]$.

Functions in $\mathrm{BLSC}_+[0,1]$ are well approximated from below by continuous functions. Then it makes sense to define, for every $f \in \mathrm{BLSC}_+[0,1]$,

$$\widetilde{I}(f) := \sup\{ I(g) \colon g \in C[0,1] \text{ and } 0 \leqslant g \leqslant f \}.$$

The properties of the extension

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(vii) If f and $f_1, f_2, \ldots \in \mathrm{BLSC}_+[0,1]$ are such that

$$f \leqslant \sum_{n=1}^{\infty} f_n \text{ then } \widetilde{I}(f) \leqslant \sum_{n=1}^{\infty} \widetilde{I}(f_n)$$

(countable sub-additivity).

Towards defining the measure

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Note: In fact we can show (homework) that

$$\mu_0(U) = \sup\{I(g) : g \in C[0,1],\ 0 \leqslant g \leqslant 1 \text{ and } \operatorname{supp}(g) \subset U\},$$

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- If $U_1 \subset U_2$ then $\mu_0(U_1) \leqslant \mu_0(U_2)$.
- If $U = \bigcup_{n=1}^{\infty} U_n$ then $\mu_0(U) \leqslant \sum_{n=1}^{\infty} \mu_0(U_n)$.

The last property follows from the countable sub-additivity of the extension $\widetilde{I}$.

Defining the outer measure

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Now for every set $E \subset [0, 1]$ define

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At last, the promised measure

By Carathéodory's extension theorem, given the outer measure $\mu^\star$, the family of sets $A \subset [0, 1]$ with the property

$$\mu^\star(E) = \mu^\star(E \cap A) + \mu^\star(E \setminus A) \quad \text{for all} \quad E \subset [0, 1]$$

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Moreover, restricted to this σ-algebra, the outer measure $\mu^\star$ is a measure, which we shall denote by μ.

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Finally, we have to check that this measure does the job, that is, for all $f \in C[0, 1]$ we have $I(f) = \int_{[0,1]} f \, d\mu$.
This follows from the fact that continuous functions can be approximated by simple functions of the form $\sum_{i=1}^{k} c_i \, \mathbb{1}_{E_i}$ with $E_1, \ldots, E_k$ open or closed sets.