The Lebesgue-Radon-Nikodym theorem

MA3105 Advanced Real Analysis

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Mutually singular measures

Let $(\Omega, \mathcal{F})$ be a measurable space.

Two signed measures μ_1 and μ_2 are called mutually singular, which we denote by

$$\mu_1 \perp \mu_2,$$

if there is an $\mathcal{F}$-measurable set E such that μ_1 is supported in E, while μ_2 is supported in $E^{\complement}$.

More precisely, for any $\mathcal{F}$-measurable set $F_1 \subset E^{\complement}$ we have $\mu_1(F_1) = 0$, while for any $\mathcal{F}$-measurable set $F_2 \subset E$ we have $\mu_2(F_2) = 0$.

Absolutely continuous measure, Radon-Nikodym derivative

Let $(\Omega, \mathcal{F})$ be a measurable space, and let m be a σ-finite, positive (reference) measure on it.

Given any measurable function $f\colon \Omega \to [0, \infty]$, the map $\mathrm{m}_f\colon \mathcal{F} \to [0, \infty]$,

$$\mathrm{m}_f(E) := \int_E f\, d\mathrm{m} = \int_\Omega \mathbb{1}_E\, f\, d\mathrm{m}$$

defines a positive measure on $(\Omega, \mathcal{F})$. Moreover, for any measurable, non-negative function g we have

$$\int_\Omega g\, d\mathrm{m}_f = \int_\Omega g f\, d\mathrm{m},$$

which may be expressed symbolically

$$d\mathrm{m}_f = f\, d\mathrm{m}$$

Similarly, if $f\colon \Omega \to \mathbb{R}$ is absolutely integrable, the corresponding map m_f is a signed, finite measure.

Absolutely continuous measure, Radon-Nikodym derivative

Let $(\Omega, \mathcal{F})$ be a measurable space, and let m be a σ-finite, positive (reference) measure on it.

Given another measure μ (signed or unsigned) on $(\Omega, \mathcal{F})$, we say that μ is absolutely continuous (or differentiable) w.r.t. m if there is a measurable function f on Ω such that

$$\mu = \mathrm{m}_f \quad \text{or symbolically,} \quad d\mu = f\, d\mathrm{m}.$$

The function f above is called the Radon-Nikodym derivative of μ w.r.t. m and we write symbolically

$$f = \frac{d\mu}{d\mathrm{m}}.$$

We know that such a function if it exists, it is unique, in the sense that if $\mathrm{m}_{f_1} = \mathrm{m}_{f_2}$, then $f_1(x) = f_2(x)$ for m-a.e. $x \in \Omega$. Hence the Radon-Nikodym derivative is well defined.

Theorem (Lebesgue-Radon-Nikodym)

Let $(\Omega, \mathcal{F}, \mathrm{m})$ be a σ-finite measure space and let μ be a σ-finite signed measure.

There exists a unique decomposition

$$\mu = \mathrm{m}_f + \mu_s$$

where $f \colon \Omega \to \mathbb{R}$ is measurable and $\mu_s \perp m$.

Moreover,

if μ is positive, then $f \geqslant 0$ m-a.e. and μ_s is positive;

if μ is finite, the $f \in L^1(\mathrm{m})$ and μ_s is finite.

Theorem (Lebesgue-Radon-Nikodym)

Let $(\Omega, \mathcal{F}, m)$ be a σ-finite measure space and let μ be a σ-finite signed measure.

The following statements are equivalent:

(1) $\mu = m_f$ for some f measurable (meaning μ is absolutely continuous w.r.t. m);

(2) For any $\mathcal{F}$-measurable set E, if $m(E) = 0$ then $\mu(E) = 0$.

(3) For any $\epsilon > 0$ there is $\delta > 0$ such that if E is an $\mathcal{F}$-measurable set with $m(E) \leqslant \delta$ then $|\mu|(E) < \epsilon$.

This theorem, namely item (3), justifies the use of the terminology μ is absolutely continuous w.r.t. m. We represent this symbolically by

$$\mu \ll m.$$

Theorem (Lebesgue decomposition)

Let $(\Omega, \mathcal{F}, m)$ be a σ-finite measure space and let μ be a σ-finite signed measure.

There exists a unique decomposition

$$\mu = \mu_{ac} + \mu_s$$

where $\mu_{ac} \ll m$ and $\mu_s \perp m$.
Moreover, if $\mu \geqslant 0$ then $\mu_{ac}, \mu_s \geqslant 0$.

If every singleton $\{x\}$, $x \in \Omega$ is a measurable set, then the singular part can be further decomposed.
Assume that m is **continuous**, meaning that $m(\{x\}) = 0$ for every $x \in \Omega$. Then there is a unique decomposition

$$\mu = \mu_{ac} + \mu_{sc} + \mu_{pp},$$

where μ_{ac} is absolutely continuous w.r.t. m, μ_{sc} is singular w.r.t. m and continuous, while μ_{pp} is a pure point measure (meaning a sum of Dirac measures).