HOMEWORK 8
THE LAW OF LARGE NUMBERS

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

Problem 1. Show that if X is a random variable such that for some positive numbers M, ϵ, δ we have

$$|X| \leq M \text{ a.s.} \quad \text{and} \quad \mathbb{P}\big(\,|X| \geq \epsilon\big) \leq \delta,$$

then

$$\mathbb{E}\,|X| \leq \epsilon + M\,\delta.$$

Problem 2. (The bounded convergence theorem (BCT) in *probability*.)

Let $X_1, X_2, \ldots$ be a sequence of random variables such that

$$\forall n \geq 1,\ |X_n| \leq M \text{ a.s.} \quad \text{and} \quad X_n \to X \text{ in probability.}$$

Then

(a) $|X| \leq M$ a.s.
(b) $\mathbb{E}X_n \to \mathbb{E}X$.

Problem 3. (Fatou's lemma for convergence in *probability*.)

Let $X_1, X_2, \ldots$ be a sequence of random variables such that

$$\forall n \geq 1,\ X_n \geq 0 \text{ a.s.} \quad \text{and} \quad X_n \to X \text{ in probability.}$$

Then

$$\mathbb{E}X \leq \liminf_{n\to\infty} \mathbb{E}X_n.$$

Problem 4. (The dominated convergence theorem (DCT) in *probability*.)

Let $X_1, X_2, \ldots$ be a sequence of random variables such that for some absolutely integrable random variable Y,

$$\forall n \geq 1,\ |X_n| \leq Y \text{ a.s.} \quad \text{and} \quad X_n \to X \text{ in probability.}$$

Then

$$\mathbb{E}X_n \to \mathbb{E}X.$$

Problem 5. Let $X_1, X_2, \ldots$ be a sequence of random variables such that $X_n \to X$ in probability. Show that if Y is another random variable such that Y is independent of *each* X_n, then Y is independent of X.

Problem 6. (Approximation of the cumulative distribution function of a random variable.)

Let $X_1, X_2, \dots$ be i.i.d. copies of a real valued random variable X. Show that for every real number t the following holds almost surely:

$$\frac{1}{n}\,\mathrm{card}\,\{1 \le i \le n \colon X_i \le t\} \to \mathbb{P}(X \le t) \quad \text{as} \quad n \to \infty.$$

Problem 7. Show that if $X_1, X_2, \dots, X_n$ are i.i.d. copies of a real valued random variable X and if $S_n := X_1 + \dots + X_n$, then

$$\mathrm{Var}\,\Big(\frac{S_n}{n}\Big) \le \frac{\mathbb{E}X^2}{n}.$$

Problem 8. Show that if $X_1, X_2, \dots, X_n$ are i.i.d. copies of a real valued random variable X, and if M is finite constant, then the truncations

$$X_1\,\mathbf{1}_{|X_1| \le M},\; X_2\,\mathbf{1}_{|X_2| \le M},\; \dots,\; X_n\,\mathbf{1}_{|X_n| \le M}$$

are also independent and identically distributed random variables.

Problem 9. Let $X_1, X_2, \dots$ be i.i.d. copies of a real valued random variable X. Prove that if $X \ge 0$ a.s. and

$$\mathbb{E}X = \infty$$

then almost surely, $\frac{S_n}{n}$ diverges to infinity in probability, in the sense that for every $T < \infty$,

$$\mathbb{P}\Big(\frac{S_n}{n} \ge T\Big) \to 1 \quad \text{as} \quad n \to \infty.$$

Hint: Truncate and use the weak LLN for the truncations. The truncation will have to be chosen carefully (you may need the monotone convergence theorem for that).

Problem 10. (The law of large numbers for triangular arrays).

Let $X_{i,n}$, where $n \ge 1$ and $1 \le i \le n$ be a triangular array of random variables with the same mean μ. Assume that each row $X_{1,1}, \dots, X_{n,n}$ consists of independent random variables and let $S_n := X_{1,1} + \dots + X_{n,n}$. Let M be a finite constant. Prove the following:

(a) (weak LLN) If $\mathbb{E}\,|X_{i,n}|^2 \le M$ a.s. for all indices, then $\frac{S_n}{n} \to \mu$ in probability.

(b) (strong LLN) If $\mathbb{E}\,|X_{i,n}|^4 \le M$ a.s. for all indices, then that $\frac{S_n}{n} \to \mu$ a.s.