HOMEWORK 6
EXPECTATION AND DISTRIBUTIONS

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

Problem 1. Prove the Cauchy-Schwarz inequality: if X, Y are non-negative random variables, then

$$\mathbb{E}\, XY \le (\mathbb{E}X^2)^{1/2}\, (\mathbb{E}Y^2)^{1/2}.$$

Moreover, if X, Y are absolutely integrable random variables, then

$$\big|\mathbb{E}\, XY\big| \le (\mathbb{E}X^2)^{1/2}\, (\mathbb{E}Y^2)^{1/2}.$$

Deduce that for any random variable X we have

$$\mathbb{E}\big|X\big| \le (\mathbb{E}\big|X\big|^2)^{1/2}.$$

Hint: Begin with the case when both X and Y are simple functions. It is convenient to represent them as

$$X = \sum_{i=1}^k a_i\, \mathbf{1}_{E_i} \quad \text{and} \quad Y = \sum_{i=1}^k b_i\, \mathbf{1}_{E_i},$$

that is, using the *same* sets $E_1, \dots, E_k$. Why is this possible?

Problem 2. Prove that if X is a random variable such that $0 < \mathbb{E}\big|X\big|^2 < \infty$ then

$$\mathbb{P}(X \neq 0) \ge \frac{(\mathbb{E}\big|X\big|)^2}{\mathbb{E}\big|X\big|^2}.$$

Problem 3. Define the L^∞ norm of a random variable X as

$$\|X\|_\infty := \inf\{M\colon \big|X\big| \le M \text{ a.s.}\}.$$

Show that if X, Y are non-negative random variables, then

$$\mathbb{E}\, XY \le \mathbb{E}\big|X\big|\, \|Y\|_\infty.$$

Problem 4. Let X be a random variable and let $F_X(x) = \mathbb{P}(X \le x)$ be its CDF.

Show that if F_X is continuous then the random variable $Y := F_X(X)$ has a uniform distribution on $(0, 1)$. In other words, if $y \in [0, 1]$, then $\mathbb{P}(Y \le y) = y$.

Problem 5. Suppose the random variable X has probability density function f.
Compute the distribution function of X^2 and then differentiate to find its density function.