HOMEWORK 4
THE RIESZ-MARKOV-KAKUTANI REPRESENTATION THEOREM

Problem 1. Let (X, d) be a metric space, let $A \subset X$ and define the function $d_A \colon X \to \mathbb{R}$ by
$$d_A(x) := \operatorname{dist}(x, A) = \inf \{ d(x, a) \colon a \in A \}.$$

Show the following:

(a) Let $L \subset X$ be a closed set. Then
$$d_L(x) = 0 \iff x \in L.$$

(b) For any $A \subset X$, the function d_A is continuous.

Problem 2. Let K be a compact subset of $\mathbb{R}$ and let U be an open subset of $\mathbb{R}$ such that
$$K \subset U.$$

Prove the following refinement of Urysohn's lemma: there is *continuous* function $f \colon \mathbb{R} \to \mathbb{R}$, with compact support, such that
$$f(x) = 0 \;\text{ if } x \in K \quad \text{ and } \quad \operatorname{supp}(f) \subset U,$$
where supp (f) is the *topological support* of f.

Hint: Let $L := U^{\complement}$, which is a closed set.

It would *not* be enough to apply Urysohn's lemma to K and L. You should instead apply it to K and to a slightly increased version of L.

Let $I \colon C[0, 1] \to \mathbb{R}$ be a positive linear functional.

We denote by $\text{BLSC}_+[0, 1]$ the family of all bounded, *lower semi continuous* functions $f \colon [0, 1] \to \mathbb{R}$ with $f \geq 0$.

For every $f \in \text{BLSC}_+[0, 1]$ we define the extension
$$\widetilde{I}(f) := \sup \{ I(g) \colon g \in C[0, 1] \text{ and } 0 \leq g \leq f \}.$$

Since for every open set $U \subset [0, 1]$, the function $\mathbf{1}_U \in \text{BLSC}_+[0, 1]$, we may then define
$$\mu_0(U) := \widetilde{I}(\mathbf{1}_U).$$

Problem 3. Show that
$$\mu_0(U) = \sup \{ I(g) \colon g \in C[0, 1],\, 0 \leq g \leq 1 \text{ and } \operatorname{supp}(g) \subset U \}.$$

Hint: Use the result in Problem 2.

Problem 4. Prove that for any $f \in \mathrm{BLSC}_+[0,1]$ we have
$$\widetilde{I}(f) \leq I(\mathbf{1})\,\|f\|,$$
where $\mathbf{1}$ denotes the constant function taking the value 1.

Problem 5. Let $f_1, f_2, \ldots \in \mathrm{BLSC}_+[0,1]$ be a sequence of functions such that $f_n \to f$, where f is continuous and $f_n \geq f$ for all n. Show that
$$\widetilde{I}(f_n) \to I(f).$$