Exercise. Let $T\colon (X, \mathcal{B}, m) \to (X, \mathcal{B}, m)$ be a measure-preserving map, where $m(X) = 1$. We say that T is *ergodic* if $T^{-1}B = B$ implies $m(B) = 0$ or $m(B) = 1$ ($B \in \mathcal{B}$). Show that the following are equivalent:

(i) T is ergodic

(ii) The only members B of $\mathcal{B}$ with $m(T^{-1}B \triangle B) = 0$ are those with $m(B) = 0$ or $m(B) = 1$. (Here $\triangle$ denotes symmetric difference.)

(iii) For every $A \in \mathcal{B}$ with $m(A) > 0$ we have

$$m\Big(\bigcup_{n=1}^{\infty} T^{-1}A\Big) = 1.$$

(iv) For every $A, B \in \mathcal{B}$ with $m(A) > 0$, $m(B) > 0$ there exists $n > 0$ with $m(T^{-n}A \cap B) > 0$.

(*Hint*: Prove (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i). The most difficult part is to show (i) $\Rightarrow$ (ii).)