Norwegian University of Science and Technology
Department of Mathematical Sciences

Contact during the mid-term exam:
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MID-TERM EXAM IN MA3105 ADVANCED REAL ANALYSIS

Monday February 25, 2008
Time: 10:15 – 12:00
Place: Room 734, SBII
Grades: Monday March 17, 2008
Permitted aids: All aids permitted.

Problem 1

Let $\mathcal{P}(\mathbb{R})$ denote the family of all subsets of $\mathbb{R}$, and let μ and ν be two measures $\mu, \nu : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ defined respectively by

$$\begin{array}{rcl} \mu(A) & = & \sharp(A \cap \mathbb{Q}) \\ \nu(A) & = & \sum_{r_n \in A} 2^{-n} \end{array}, \quad \text{der } A \subset \mathbb{R}$$

(Here $\mathbb{Q} = \{r_1, r_2, \cdots r_n, \cdots\}$ is the set of rational numbers.)

a) Show that both μ and ν are σ-finite and complete measures.

b) Let $\overline{\mu \times \nu}$ denote the completion of the product measure $\mu \times \nu$, and let $(\mathbb{R} \times \mathbb{R}, \mathcal{M}, \overline{\mu \times \nu})$ be the associated measure space. Show that $\mathcal{M} = \mathcal{P}(\mathbb{R} \times \mathbb{R})$, and give an explicit formula for $\overline{\mu \times \nu}(B)$, (in terms of $r_1, r_2, \cdots, r_n, \cdots,$) where $B \subset \mathbb{R} \times \mathbb{R}$.

c) Let $f = \chi_\Delta$, where $\Delta = \{(x, x) | x \in \mathbb{R}\}$ is the diagonal set in $\mathbb{R} \times \mathbb{R}$. Compute the three integrals

$$\int_{\mathbb{R}} (\int_{\mathbb{R}} f(x, y) d\mu(x)) d\nu(y), \quad \int_{\mathbb{R}} (\int_{\mathbb{R}} f(x, y) d\nu(y)) d\mu(x), \int_{\mathbb{R} \times \mathbb{R}} f d\overline{\mu \times \nu}$$

explicitly, and explain why you get the same value.

Problem 2

Define $f_n\colon \mathbb{R} \to \mathbb{R}$ by the formula

$$f_n(x) = \frac{n}{\pi(1+n^2x^2)} \quad , n = 1, 2, \cdots$$

a) Show that

(i) $\int_{-\infty}^{\infty} f_n(x)dx = 1$ for all n.

(ii) For every $\delta > 0$ we have $\displaystyle\int_{|x|>\delta} f_n(x)dx \to 0$ as $n \to \infty$.

b) Let $g \in L^1(\mathbb{R})$. Show that for each n the function

$$(g * f_n(x) =) f_n * g(x) = \int_{-\infty}^{\infty} f_n(x-y)g(y)dy$$

is uniformly continuous on $\mathbb{R}$. Show also that $f_n * g$ is differentiable, i.e. the derivative exists for each $x \in \mathbb{R}$.

c) Show that $f_n * g$ converges to g in L^1-norm as $n \to \infty$, i.e.

$$\int_{-\infty}^{\infty} |f_n * g(x) - g(x)|dx \to 0 \text{ når } n \to \infty.$$

Problem 3

Let $(\mathbb{R}^d, \mathcal{M}, m)$ be the Lebesgue measure space in $\mathbb{R}^d$. Define a new measure μ on $\mathcal{M}$ by

$$\mu(E) = \int_E (1+|x|^2)dx, \quad E \in \mathcal{M}.$$

a) Show that if $f : \mathbb{R}^d \to \mathbb{C}$ is integrable with respect to μ, then

$$\int_{\mathbb{R}^d} f(x)d\mu(x) = \int_{\mathbb{R}^d} (1+|x|^2)f(x)dm(x)$$

b) Show that the measure μ is *regular*, i.e. for every $E \in \mathcal{M}$ and $\in > 0$, there exists an open set $\mathcal{O}$ in $\mathbb{R}^d$ such that $E \subset \mathcal{O}$ and $\mu(\mathcal{O}) \leq \mu(E) + \in$.

[Hint: Consider $E_n = E \cap B_n(0), n = 1, 2, \cdots$, where $B_n(0) = \{x \in \mathbb{R}^d \| x| \leq n\}$.]

Problem 4 (Can be dropped).

Prove the Poisson summation formula:

$$(*) \qquad \sum_{n=-\infty}^{\infty} f(x+n) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n x}$$

where $f \in \mathcal{S}(\mathbb{R})$. In particular, setting $x = 0$, we get:

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n)$$

[Hint: Prove first that both sides in $(*)$ are continuous functions of x. Then observe that both sides are periodic functions of x with period 1. Compute the Fourier coefficients in the interval $[0,1]$ of the two sides in $(*)$. (The n'th Fourier coefficient of an integrable function $g : [0,1] \to \mathbb{C}$ is:

$$\int_0^1 g(t) e^{-2\pi i n t} dt).]$$