**Problem 1** The set $\{X, \emptyset, A\}$ contains $X$ and $\emptyset$, and is closed under arbitrary unions and intersections. Therefore it is a topology on $X$. It is in fact the coarsest topology on $X$ for which the set $A$ is an open set. ∎

**Problem 2** The partially ordered set $\{X, \leq\}$ is inductive if every totally (linearly) ordered subset is dominated by an element of $X$.

$$\forall L(L \text{ totally ordered subset of } X \;\Rightarrow \exists x(x \in X \land \forall y(y \in L \Rightarrow y \leq x)))$$

Zorn's Lemma says that any inductive partially ordered set has an element that is maximal.

$$X \text{ inductive} \Rightarrow \exists x(x \in X \land \forall y(y \in X \land x \leq y \Rightarrow x = y))$$

**Problem 3** Let $B_i = A \cup A_i$. Then we have $C = A \cup \bigcup_{i \in I} A_i = \bigcup_{i \in I} B_i$. The set $B_i$ is connected since both $A$ and $A_i$ are connected and have something in common. Finally $C$ is connected since $A$ is nonempty and contained in each $B_i$. ∎

**Problem 4** We divide the square $I \times I$ into three regions.

$$\begin{aligned} I &= \{(s,t) \,|\, 0 \leq s \leq 1, \quad 0 \leq t \leq 1/2 - s/2\}, \\ II &= \{(s,t) \,|\, 0 \leq s \leq 1, \quad 1/2 - s/2 \leq t \leq 1/2 + s/2\}, \\ III &= \{(s,t) \,|\, 0 \leq s \leq 1, \quad 1/2 + s/2 \leq t \leq 1\}. \end{aligned}$$

A possible homotopy is

$$F(s,t) = \begin{cases} f(2t) & \text{for } (s,t) \in I, \\ f(1-s) & \text{for } (s,t) \in II, \\ f(2-2t) & \text{for } (s,t) \in III. \end{cases}$$

This was the "stop and wait" homopopy. A simpler solution is this.

$$F(s,t) = \begin{cases} f((1-s)2t) & \text{for} \quad 0 \leq t \leq 1/2, \\ f((1-s)(2-2t)) & \text{for} \quad 1/2 \leq t \leq 1. \end{cases}$$

**Problem 5** We write $X_i$ for the topological space $(X, \mathcal{T}_i)$, $i \in \{1,2\}$. We may assume that $\mathcal{T}_1 \subseteq \mathcal{T}_2$, which means that the identity map $id : X_2 \to X_1$ is continuous. Since $X_2$ is compact, a closed set $F \subseteq X_2$ is compact. The continuous image of a compact set is compact, so $F$ is a compact subset of the Hausdorff space $X_1$. Any compact subset of a Hausdorff space is closed so in fact $F$ is a closed subset of $X_1$. The identity map takes complements to complements (in fact it does nothing). This shows that open subsets of $X_2$ are open in $X_1$, which means that $\mathcal{T}_2 \subseteq \mathcal{T}_1$, and so $\mathcal{T}_1 = \mathcal{T}_2$. ∎

**Problem 6** We say a subset $A \subseteq X$ is $f$-invariant if the following holds. $(a \in A \wedge f(a){=}f(b)) \Rightarrow b \in A$. The $f$-invariant subsets of $X$ are of the form $f^{-1}(B)$ for some $B \subseteq Y$. Since $f$ is surjective, this set $B$ is unique and $B = f(A)$. This means that for a surjective map $f$, the inverse image $f^{-1}$ defines a bijective map between the $f$-invariant subsets of $X$ and all subsets of $Y$. It follows that $f$ preserves complements. This means that if $A \subseteq X$ is $f$-invariant, then $f(X - A) = Y - f(A)$. The same reasoning as in Problem 5 shows that $f$ maps closed sets to closed sets, and hence it maps $f$-invariant open sets to open sets.
If $U \subseteq Z$ is any open subset of $Z$ then since $g \circ f$ is continuous, $V = (g \circ f)^{-1}(U)$ is both open and $f$-invariant, and hence $f(V) = g^{-1}(U)$ is open in $Y$. This shows that $g$ is continuous. ■

**Problem 7** The space $S^1 \times I$ is compact and the disc $D$ is Hausdorff. Since the map $f$ is surjective it follows from Problem 6 that $D$ is a quotient space. ■

**Problem 8a** The open sets of $X/A$ are exactly the $\pi$-invariant open sets of $X$. A $\pi$-invariant subset of $X$ is any set containing $A$ or disjoint from $A$. Since $F$ is closed and $\pi$-invariant, its complement is open and $\pi$-invariant. This shows that $\pi(F)$ is closed. ■

**Problem 8b** Points are closed in a Hausdorff space, so the singleton $\{A\} \subseteq X/A$ is closed. Taking the inverse image of its complement shows that $X - A$ is open in $X$, and therefore $A$ is closed. ■

**Problem 9a** Since $\mathbb{N}$ is given the discrete topology, a neighborhood of a point $(z, n) \in X$ is any set containing a set of the form $I \times \{n\}$ where $I$ is an open interval of $S^1$ containing $z$. This shows that $F$ is closed in $X$. Moreover $F \cap A = \emptyset$, so by Problem 8a, $\pi(F)$ is closed in $X/A$. ■

**Problem 9b** If $K$ is compact, since $\pi(F)$ is closed, it follows that $K \cap \pi(F)$ is compact. Since $\mathbb{N}$ is given the discrete topology, a neighborhood of a point $(z, n)$ where $z \neq 1$ is any set containing a set of the form $I \times \{n\}$ where $I$ is an open interval of $S^1$ containing $z$ but not 1. This shows that $\pi(F)$ is discrete. A subspace of a discrete space is also discrete, therefore $K \cap \pi(F)$ is both discrete and compact. Discrete compact spaces are finite. ■