Norwegian University of Science and Technology
Department of Mathematical Sciences

Contact under the exam:
Finn Faye Knudsen tlf. 73 59 35 23

EXAM IN MA3002 GENERAL TOPOLOGY

English
Tuesday May 23 2006
09:00–13:00

Aids (code D): No written or printed aids permitted.
Simple calculator (HP 30S)

Grades are ready: June 15 2006.

Problem 1 Let X be a set and let $A \subseteq X$. Is $\{\emptyset, A, X\}$ a topology on X?

Problem 2 Explain what is meant by "the partially ordered set $(X, \leq)$ is *inductive*". Explain what it means that an element $x \in X$ is *maximal*. State Zorn's Lemma.

Problem 3 Let $\{A_i\}_{i\in I}$ be an indexed family of connected subspaces of a space X, and let A be a connected subspace of X. Show that if $A \cap A_i \neq \emptyset$ for all $i \in I$, then $A \cup (\bigcup_{i\in I} A_i)$ is connected.

Problem 4 Let I be the unit interval and let X be a space. Let $f : I \to X$ be a loop based at a point $x_0 \in X$. Let $g : I \to X$ be the loop defined by $g(t) = f(1 - t)$. Construct an explicit path homotopy from $g * f$ to the constant loop e.

Recall that the loop $g * f$ is defined by

$$(g * f)(t) = \begin{cases} f(2t) & \text{for} \quad 0 \leq t \leq \frac{1}{2} \\ g(2t-1) & \text{for} \quad \frac{1}{2} \leq t \leq 1. \end{cases}$$

Problem 5 Let X be a set and let $\mathcal{H}$ be the family of all topologies on X that make X a Hausdorff space, and let $\mathcal{C}$ be the family of all topologies on X that make X a compact space. Show that if T_1 and T_2 are comparable topologies in $\mathcal{H} \cap \mathcal{C}$, then $T_1 = T_2$.

Problem 6 Let $f : X \to Y$ be a continuous surjection from a compact space X to a Hausdorff space Y. Let $g : Y \to Z$ be an arbitrary map. Show that if $g \circ f$ is continuous, then g is continuous.

Problem 7 Let $S^1 = \{z \mid z \in \mathbb{C} \wedge |z| = 1\}$ be the unit circle, $D = \{z \mid z \in \mathbb{C} \wedge |z| \leq 1\}$ be the closed unit disc, and let $I = [0, 1]$ be the closed unit interval. Show that the map $f : S^1 \times I \to D$ defined by $f(z, t) = tz$ is a quotient map.

Problem 8 Let $A \subseteq X$ be an arbitrary subset of a topological space X. Let $X/A = (X - A) \cup \{A\}$. Note that $\{A\}$ is a set with exactly one element, the set A. We define the map $\pi : X \to X/A$ by

$$\pi(x) = \begin{cases} x & \text{for } x \notin A \\ A & \text{for } x \in A. \end{cases}$$

We give X/A the quotient topology.

a) Show that if $F \subseteq X$ is closed and $F \cap A = \emptyset$, then $\pi(F)$ is closed in X/A.

b) Show that if X/A is Hausdorff, then the set A is closed in X.

Problem 9 In the following we let S^1 be the unit circle, and $\mathbb{N}$ be the natural numbers with the discrete topology. Let $X = S^1 \times \mathbb{N}$ and let $A = \{1\} \times \mathbb{N}$. The space X/A is constructed as in Problem 8. It is the space obtained from X by letting A shrink to a single point.

Let $\{z_n\}_{n \in \mathbb{N}}$ be a sequence of points in $S^1 - \{1\}$. This means that $z_n \neq 1$ for all $n \in \mathbb{N}$.

a) Let $F = \{(z_n, n) \mid n \in \mathbb{N}\}$. Show that the set $\pi(F)$ is closed in X/A.

b) Let $K \subseteq X/A$ be a compact subset of X/A. Show that the set $K \cap \pi(F)$ is finite.

Hint: What is the topology of the subspace $\pi(F)$ of X/A?