The spectrum of

$$SH(G)^c$$

The spectrum of the stable hmtpy cat of G-equivariant spectra

$\operatorname{spc} SH(G)^c = ?$

First recall some facts:

Abuse of notation write

$$G/K_+ \text{ for } \Sigma^\infty_G G/K_+$$

$$\left\{ G/K_+ \;\middle|\; K \leq G \right\}$$

form compact generators for $SH(G)$.

Group hom $G' \xrightarrow{\alpha} G$ induces

$$SH(G) \xrightarrow{\alpha^*} SH(G')$$

if $G' \leq G$ we call α^* $res^G_{G'}$

if $G = G'/N$ we call α^* infl

if $G = 1$ we call α^* triv

Recall geometric fixed points

$$\mathcal{L}_N := \langle G/K_+ \mid N \not\subseteq K \rangle$$

$$\widetilde{\Phi}^N : SH(G) \to SH(G)/\mathcal{L}_N \simeq SH(G/N)$$

$$\begin{tikzcd}
SH(G) \arrow[r, dashed, "\Phi^H"] \arrow[d, "\mathrm{res}"] & SH \arrow[d, phantom, "\cong" rotate=90] \\
SH(H) \arrow[r, "\widetilde{\Phi}^H"] & SH(H/H)
\end{tikzcd}$$

$$\Phi^H(\Sigma^\infty_G X_+) = \Sigma^\infty X^H_+$$

$$G' \xrightarrow{\alpha} G \qquad\qquad K \leq H \leq G$$

$$\rightsquigarrow \qquad\qquad\qquad\qquad \rightsquigarrow$$

$$\begin{tikzcd}
SH(G) \arrow[r, "\phi^{\alpha(H)}"] \arrow[d] & SH \\
SH(G') \arrow[ur, "\phi^{H}"'] \arrow[u, phantom, shift left=6ex, "\circ"]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
SH(G) \arrow[r, "\phi^{K}"] \arrow[d, "res"] & SH \\
SH(H) \arrow[ur, "\phi^{K}"'] \arrow[u, phantom, shift left=6ex, "\circ"]
\end{tikzcd}$$

Recall $\mathrm{spc}\, SH^c$ is given by

$$\begin{tikzcd}[column sep=small]
e_{2,\infty} \arrow[d, dotted, no head] & e_{3,\infty} \arrow[d, dotted, no head] & & e_{p,\infty} \arrow[d, dotted, no head] & \\
e_{2,3} \arrow[d, no head] & e_{3,3} \arrow[d, no head] & & e_{p,n} \arrow[d, dotted, no head] & \\
e_{2,2} \arrow[dr, no head] & e_{3,2} \arrow[d, no head] & \cdots & e_{p,2} \arrow[dll, no head] & \cdots \\
& e_{0,1} & & &
\end{tikzcd}$$

<u>Thm 4.9</u>: All primes in $\mathrm{spc}\,SH(G)^c$ are given by

$$\mathcal{P}(H,p,n) := (\Phi^H)^{-1}(\mathcal{C}_{p,n})$$

$$\left\ulcorner (\Phi^H)^{-1} = \mathrm{spc}\,\Phi^H =: \varphi^{H,G} \right\lrcorner$$

<u>proof</u>: Induction on $|G|$.

$\hookrightarrow$ $|G| = 1$ $SH(G) = SH$ and $\Phi^G = \mathrm{id}$, so nothing to show. ✓

Note: we want to prove that

$$\mathrm{src}\, SH(G)^c = \bigcup_{H \leq G} \mathrm{Im}\, g^{H,G}$$

because

$$P(H, p, n) = g^{H,G}(C_{p,n})$$

It's a general fact that when K is tt $J \subseteq K$ is loc/~~t~~thick then

$$\mathrm{spc}\left(K/J\right) = \{ \mathcal{P} \in \mathrm{spc}\, K \mid J \subseteq \mathcal{P} \}$$

Since $\Phi^G : SH(G) \to SH(G)/\mathcal{L}_G$ is localization with $\mathcal{L}_G = \langle G/K_+ \mid K \subsetneq G \rangle$

$$\mathrm{Im}\, \varphi^{G,G} = \{ \mathcal{P} \mid G/K_+ \in \mathcal{P}\ \forall K \subsetneq G \}$$

We have

$$\operatorname{src} SH(G)^c = \operatorname{Im} \varphi^{G,G} \cup \bigcup_{H \subsetneq G} \operatorname{supp}(G/H_+)$$

because

$$\operatorname{supp}(G/H_+) = \{ \mathfrak{p} \mid G/H_+ \notin \mathfrak{p} \}$$

and each prime either does or does not contain G/H_+.

Lemma (Black magic)

tt

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, shift left, "U"] \end{tikzcd}$$

adjoint

$$\Rightarrow \quad \operatorname{Im} \operatorname{spc} F = \operatorname{Supp}(U(\mathbb{1}_{\mathcal{D}}))$$

$$\begin{tikzcd} SH(G) \arrow[r, shift left, "\mathrm{res}"] \arrow[r, phantom, "\perp"] & SH(H) \arrow[l, shift left, "\mathrm{coind}"] \end{tikzcd}$$

$$\begin{array}{ccc} \mathrm{Map}^G(G \times_H H/H, X) & = & \mathrm{Map}^H(H/H, \mathrm{res}\,X) \\ \| & & \| \\ \mathrm{Map}(G/H_+, X) & = & \mathrm{Map}^H(\mathbb{1}_H, \mathrm{res}\,X) \end{array}$$

$$\Rightarrow \operatorname{Im} \operatorname{spc} \operatorname{res}^G_H = \operatorname{Supp}(G/H_+)$$

Induction $\Rightarrow$ $\operatorname{spc} SH(H)^c$

$$= \bigcup_{K \leq H} \operatorname{Im} \varphi^{K,H}$$

$$\Rightarrow \operatorname{Supp}(G/H_+) = \bigcup_{K \leq H} \operatorname{Im} \operatorname{spc} \operatorname{res} \circ \varphi^{K,H}$$

$$\begin{tikzcd}
SH(G) \arrow[dr, "\phi^K"] \arrow[d, "\mathrm{res}"'] & \\
SH(H) \arrow[r, "\phi^K"'] & SH
\end{tikzcd}
\quad \circ$$

$$\Rightarrow \bigcup_{K \leq H} \operatorname{Im} \varphi^{K,G}$$

$=$

Conclusion

$$\operatorname{spc} SH(G)^c = \bigcup_{H \leq G} \operatorname{Im} \varphi^{H,G}$$

$\Rightarrow$ all primes of the form

$$\varphi^{H,G}(\mathcal{C}_{p,n}) = \mathcal{P}(H,p,n) \quad \blacksquare$$

Thm 4.14

$$\wp(H,p,n) = \wp(K,q,m)$$

$\Rightarrow$ $H \underset{G}{\sim} K$ $\left(\exists g \ \ g^{-1}Hg = K\right)$

and $e_{p,n} = e_{qm}$

$\left(\begin{array}{c} p = q \text{ and } , n = m \\ \text{or } \ n = m = 1 \end{array}\right)$

Group theory Lemma:

$$\left(G/K\right)^H = \{gK \mid g^{-1}Hg \subseteq K\}$$

proof:

$$g^{-1}Hg \subseteq K \Rightarrow h \cdot gK$$
$$= gg^{-1}hgK$$
$$= gK \checkmark$$

$$h \cdot gK = gK \Rightarrow hg \in gK$$
$$\Rightarrow g^{-1}hg \in K \checkmark$$

$\blacksquare$

Recall $\mathcal{P}(H,\nu,n) = (\Phi^H)^{-1}(\mathcal{C}_{\varphi,n})$

$$\Phi^H\left(\Sigma_G^\infty X_+\right) = \Sigma^\infty X^H_+$$

Cor 4.12

$$\mathcal{P}(H,\nu,n) \subseteq \mathcal{P}(K,\varphi,m)$$

$\Rightarrow \quad H \leq_G K \quad (\exists g \; g^{-1}Hg \leq K)$

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proof

$H \not\leq_G K \Rightarrow \left(G/K\right)^H = \phi$

$$\Rightarrow \quad \Phi^H\left(G/K_+\right) = \Sigma^\infty \emptyset_+ = 0\text{-object}$$

$$X := \left(G/K\right)^K \neq \emptyset$$

$$\Phi^K\left(G/K_+\right) = \Sigma^\infty X_+$$

is a wedge of $|X|$ circles. So

$\Phi^K(G/K_+) = \mathbb{1}^{\oplus |X|}$ which is not contained in any primes!

$$\Rightarrow \quad G/K_+ \in \mathcal{P}(H, p, n)$$

but $G/K_+ \notin \mathcal{P}(K, q, m)$ $\boxtimes$

Proof of 4.14

$$\mathcal{P}(H, p, n) = \mathcal{P}(K, q, m)$$

$$\Rightarrow H \leq_G K \text{ and } K \leq_G H$$

$$\Rightarrow H \sim_G K \checkmark$$

$$\Phi^H \circ \mathrm{infl} = \mathrm{id}$$

$$\Rightarrow \mathrm{spc\,infl}(\mathcal{P}(H, p, n)) = \mathcal{C}_{p,n}$$

$$\mathrm{spc\,infl}(\mathcal{P}(K, q, m)) = \mathcal{C}_{q,m}$$

$$\Rightarrow \mathcal{C}_{p,n} = \mathcal{C}_{q,m} \quad \blacksquare$$

Let $\mathcal{C}$ be the category of finite G-sets. The grothendieck group has a ring structure with multiplicat[ion] the cartesian product.

We call this ring the Burnside ring

$$A(G)$$

It is a theorem that

$$\mathrm{End}(\mathbb{1}_{SH(G)}) = A(G)$$

and Φ^H induces the map

$$\begin{aligned} \mathcal{S}^H : A(G) &\longrightarrow \mathbb{Z} \\ X &\longmapsto |X^H| \end{aligned}$$

Thm 3.6 (Dress)

All primes in spc $A(G)$ are of the form $(\mathcal{S}^H)^{-1}(p)$, but there may be collisions!!

Collisions correspond to interesting (unsolved!!) topological behavior through the comparison map.

See figure:

(1.3)