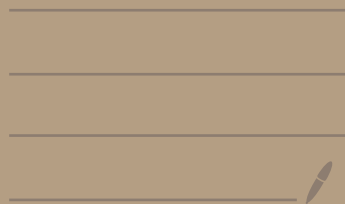


The spectrum of

$$SH(G)^c$$



The spectrum of
the stable homotopy cat
of G -equivariant
spectra

$$\mathrm{spc} \mathrm{SH}(G)^c = ?$$

First recall some
facts:

Abuse of notation write

$$G/K_+ \text{ for } \sum_G^\infty G/K_+$$

$$\left\{ G/K_+ \mid K \leq G \right\}$$

form compact generators
for $SH(G)$.

Group hom $G' \xrightarrow{\alpha} G$

induces

$$SH(G) \xrightarrow{\alpha^*} SH(G')$$

if $G' \leq G$ we call α $\text{res}_{G'}^G$

if $G = G'/N$ we call α infl

if $G = 1$ we call a triv

Recall geometric fixed points

$$L_N := \langle G/K_+ \mid N \nmid K \rangle$$

$$\varphi^N : SH(G) \rightarrow SH(G) \Big/_{L_N} \cong SH\left(\frac{G}{N}\right)$$

$$\begin{array}{ccc} \varphi^H : SH(G) & \dashrightarrow & SH \\ \downarrow \text{res} & \varphi^H & \cong \\ SH(H) & \longrightarrow & SH\left(\frac{H}{H}\right) \end{array}$$

$$\varphi^H \left(\sum_G^\infty X_+ \right) = \sum^\infty X_+^H$$

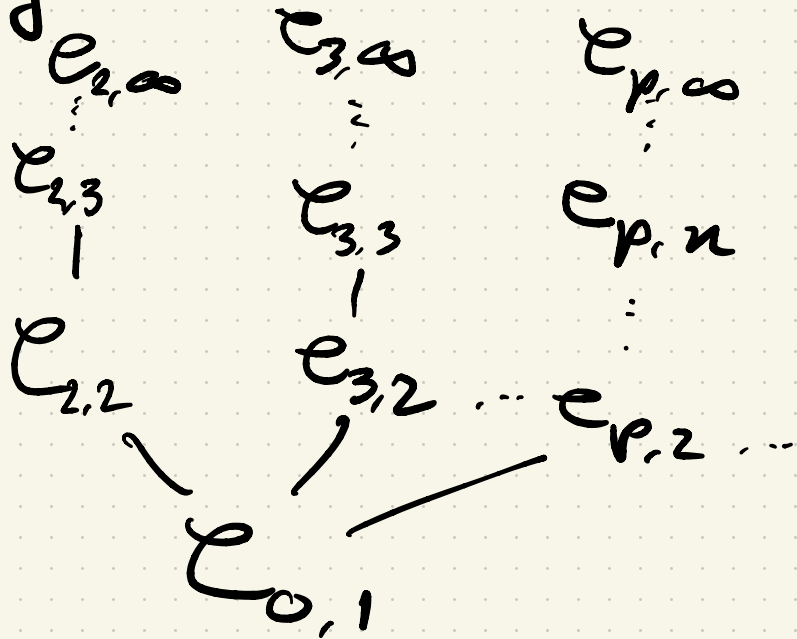
$$G' \xrightarrow{\alpha} G$$

$$K \leq H \leq G$$

$$\begin{array}{ccc} & \mathbb{E} & \\ SH(G) & \xrightarrow{\phi^{\alpha(H)}} & SH \\ \downarrow & \circlearrowleft & \uparrow \\ SH(G') & \xrightarrow{\phi^{\alpha}} & SH \end{array}$$

$$\begin{array}{ccc} & \mathbb{E} & \\ SH(G) & \xrightarrow{\phi^K} & SH \\ \downarrow \text{res} & \circlearrowleft & \uparrow \\ SH(H) & \xrightarrow{\phi^K} & SH \end{array}$$

Recall $\text{spec } SH^c$ is given by



Thm 4.9: All primes
in $\text{spec SH}(G)^c$ are given
by

$$\mathcal{P}(H, \rho, \alpha) := (\Phi^H)^{-1}(\mathcal{L}_{\rho, \alpha})$$

$$\left[(\Phi^H)^{-1} = \text{spec } \Phi^H =: \mathcal{G}^{H, G} \right]$$

proof: Induction on $|G|$.

$$\hookrightarrow |G|=1 \quad \text{SH}(G) = \text{SH}$$

and $\Phi^G = \text{id}$, so

nothing to show. ✓

Note: we want
to prove that

$$\text{spec } \mathcal{SH}(G)^c = \bigcup_{H \leq G} \text{Im } \mathcal{G}^{H, G}$$

because

$$\mathcal{P}(H, p, n) = \mathcal{G}^{H, G}(\mathcal{C}_{p, n})$$

It's a general fact
 that when K is tt
 $J \subseteq K$ is loc/quick
 then

$$\text{src}\left(\frac{K}{J}\right) = \left\{ \mathcal{P} \in \text{spec } K \mid J \subseteq \mathcal{P} \right\}$$

Since $\phi^G: \text{SH}(G) \rightarrow \frac{\text{SH}(G)}{L_G}$
 is localization

with $L_G = \left\langle \frac{G}{K_+} \mid K \not\subseteq G \right\rangle$

$$\text{Im } \mathcal{S}^{G, G} = \left\{ \mathcal{P} \mid \frac{G}{K_+} \in \mathcal{P} \forall K \not\subseteq G \right\}$$

We have

$$\text{src } \text{SH}(G)^c = \text{In } \mathcal{P}^{G, G} \cup_{H \neq G} \text{supp}\left(\frac{G}{H}\right)$$

because

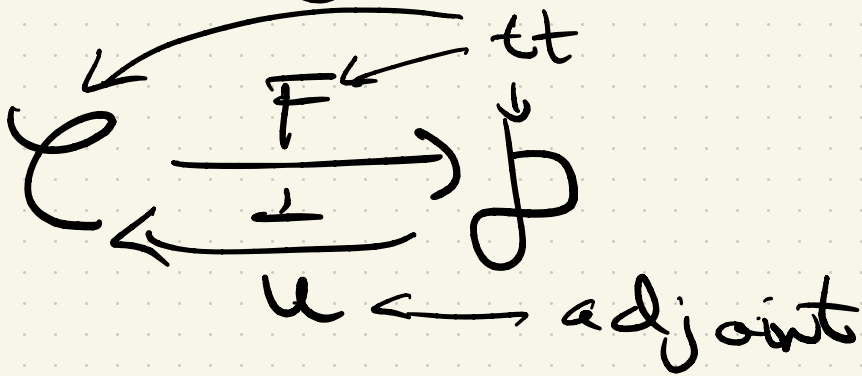
$$\text{supp}\left(\frac{G}{H}\right) = \left\{ \mathcal{P} \mid \frac{G}{H} \notin \mathcal{P} \right\}$$

and each prime

either does or does

not contain $\frac{G}{H}$.

Lemma (Black magic)



$$\Rightarrow \text{Im } \text{spec } F = \text{Supp}(U(\mathbb{1}_{\mathcal{D}}))$$

$$\text{SH}(G) \begin{array}{c} \xrightarrow{\text{res}} \\ \downarrow \perp \\ \xleftarrow{\text{coid}} \end{array} \text{SH}(H)$$

$$\text{Map}^G(G \times_{\mathbb{H}}^{\mathbb{H}}, X) = \text{Map}^H(\mathbb{H}_{\mathbb{H}}, \text{res} X)$$

||

$$\text{Map}(G_{\mathbb{H}^+}, X) = \text{Map}^H(\mathbb{I}_{\mathbb{H}}, \text{res} X)$$

$$\Rightarrow \text{Im spec ves}_H^G = \text{Supp}\left(\frac{G}{H_+}\right)$$

$$\text{Induction} \Rightarrow \text{spec SH}(H)^c$$

$$= \bigcup_{K \leq H} \text{Im } \mathcal{G}^{K, H}$$

$$\Rightarrow \text{Supp}\left(\frac{G}{H_+}\right) = \bigcup_{K \leq H} \text{Im spec ves } \mathcal{G}^{K, H}$$

$$\begin{array}{ccc}
 \text{SH}(G) & \xrightarrow{\phi^K} & \text{SH} \\
 \downarrow \text{ves} & \searrow & \nearrow \phi^K \\
 \text{SH}(H) & \circlearrowleft &
 \end{array}
 \Rightarrow \bigcup_{K \leq H} \text{Im } \mathcal{G}^{K, G}$$

$\uparrow =$

Conclusion

$$\text{spec } S_H(G)^c = \bigcup_{H \leq G} \text{In } \mathfrak{g}^{H, G}$$

\Rightarrow all primes of the form

$$\mathfrak{S}^{H, G}(\mathfrak{p}, u) = \mathcal{P}(H, r, u) \quad \text{E}$$

Thm 4.14

$$\mathcal{P}(H, p, u) = \mathcal{P}(K, q, m)$$

$$\Rightarrow H \sim_G K \quad (\exists g \quad g^{-1}Hg = K)$$

and $\mathcal{L}_{p,u} = \mathcal{L}_{q,m}$

$$\left(\begin{array}{l} p=q \text{ and } u=m \\ \text{or } u=m=1 \end{array} \right)$$

Group theory Lemma:

$$\left(\frac{G}{K}\right)^H = \{gK \mid g^{-1}Hg \subseteq K\}$$

proof:

$$g^{-1}Hg \subseteq K \Rightarrow h \cdot gK$$

$$= gg^{-1}kgK$$

$$= gK \quad \checkmark$$

$$h \cdot gK = gK \Rightarrow hge \in gK$$

$$\Rightarrow g^{-1}kg \in K \quad \checkmark$$

~~□~~

Recall $\mathcal{P}(H, v, n) = (\phi^H)^{-1}(\mathcal{C}_{p,n})$

$$\phi^H(\sum_G^\infty X_+) = \sum^\infty X_+^H$$

Cor 4.12

$$\mathcal{P}(H, v, n) \subseteq \mathcal{P}(K, q, m)$$

$$\Rightarrow H \leq_G K \quad (\exists g \quad g^{-1}Hg \leq K)$$

proof

$$H \not\leq_G K \Rightarrow \left(\frac{G}{K}\right)^H = \emptyset$$

$$\Rightarrow \phi^H\left(\frac{G}{K}_+\right) = \sum^\infty \emptyset_+ = 0\text{-object}$$

$$X := \left(\frac{G}{K}\right)^K \neq \emptyset$$

$$\Phi^K\left(\frac{G}{K_+}\right) = \sum^{\infty} X_+$$

is a wedge of $|X|$
circles. So

$\Phi^K\left(\frac{G}{K_+}\right) = \mathbb{I}^{\oplus |X|}$ which is
not contained in any
primes!

$$\Rightarrow \frac{G}{K_+} \in \mathcal{P}(H, v, u)$$

$$\text{but } \frac{G}{K_+} \notin \mathcal{P}(K, \tau, m) \quad \square$$

Proof of 4.14

$$P(H, v, u) = P(K, q, u)$$

$$\Rightarrow H \leq_G K \text{ and } K \leq_G H$$

$$\Rightarrow H \cup_G K \checkmark$$

$$\Phi^H \circ \text{in}_G = \text{id}$$

$$\Rightarrow \text{spc in}_G(P(H, v, u)) = \mathcal{C}_{v, u}$$

$$\text{spc in}_G(P(K, q, u)) = \mathcal{C}_{q, u}$$

$$\Rightarrow \mathcal{C}_{v, u} = \mathcal{C}_{q, u} \quad \square$$

Let \mathcal{C} be the category of finite G -sets. The Grothendieck group has a ring structure with multiplication the cartesian product. We call this ring the Burnside ring

$$A(G)$$

It is a theorem that

$$\text{End}(\mathbb{I}_{S^H(G)}) = A(G)$$

and ϕ^H induces the map

$$S^H: A(G) \rightarrow \mathbb{Z}$$

$$x \mapsto |x^H|$$

Thm 3.6 (Dress)

All primes in $\text{spec } A(G)$
are of the form

$(S^H)^{-1}(\mathfrak{p})$, but there

may be collisions!!

Collisions correspond to interesting (unsolved!!) topological behavior through the comparison map.

See figure:

