TALK 8

Fixed point functors in equivariant stable homotopy theory.

Introduction

Equivariant homotopy theory is the setting where we study homotopy theory with symmetry. Symmetry in mathematics is described by groups, most often Lie groups and finite groups. We restrict ourselves to finite groups for this talk.

Def: Let G be a finite group. A G-space X is a topological space together with a G-action, i.e. a map $G \times X \longrightarrow X$ such that

- $e \cdot x = x$
- $g_1(g_2(x)) = (g_1 g_2) x$.

We want a category Top_G of G-spaces. What should the maps, $\mathrm{Map}^G(X,Y)$ be?

Def: Let X, Y be G-spaces. A map $f: X \longrightarrow Y$ is G-equivariant if

$$f(g\cdot x) = g\cdot f(x) \quad \text{for all } g \in G.$$

Recall that the set of G-fixedpoints of a G-space X is the set

$$X^G = \{x \in X \mid gx = x \;\; \forall g \in G\}$$

Now, for an inclusion $H \xhookrightarrow{i} G$ we get a functor $\mathrm{Top}_G \xrightarrow{i_H^*} \mathrm{Top}_H$.

This has a left and right adjoint

$$\begin{tikzcd}
\mathrm{Top}_G \arrow[r] \arrow[r, phantom, shift left=4ex, "\perp"] \arrow[r, phantom, shift right=4ex, "\perp"] & \mathrm{Top}_H \arrow[l, bend right=60, "\mathrm{Ind}_H^G = G \times_H (-)"'] \arrow[l, bend left=60, "\mathrm{Coind}_H^G = \mathrm{Map}^G(H,-)"]
\end{tikzcd}$$

We get

$$\begin{tikzcd} \mathrm{Map}^G(G \times_H H/H, X) \arrow[r, phantom, "\overset{\text{From adjunction}}{\cong}"] \arrow[d, phantom, sloped, "\cong"] & \mathrm{Map}^H(H/H, i_H^* X) \arrow[d, phantom, sloped, "\cong"] \\ \mathrm{Map}^G(G/H, X) & X^H \end{tikzcd}$$

Now, take the projection $G \longrightarrow G/N$. We get a functor $\mathrm{Top}^{G/N} \longrightarrow \mathrm{Top}^G$ which has a right adjoint given by coinduction, i.e.

$$\begin{tikzcd} \mathrm{Top}^{G/N} \arrow[r] \arrow[r, phantom, shift right=3, "\perp"] & \mathrm{Top}^G \arrow[l, bend left=40, "\mathrm{Map}^G(G/N, -)"] \end{tikzcd}$$

Hence its adjoint gives N-fixed points, as $\mathrm{Map}^G(G/N, X) \cong X^N$.

The category $SH(G)$

Last time we converted spaces into spectra by stabilizing, i.e. inverting the suspension functor. We have a similar construction

$$\begin{tikzcd} \mathrm{Top}_G \arrow[r, squiggly] & \boxed{\begin{array}{c}\text{Homotopy theory}\\ \text{Black Box}\end{array}} \arrow[r, squiggly] & \mathrm{Sp}_G \end{tikzcd}$$

For the insistent: We can get Sp_G by inverting all representation spheres S^V, where V is a representation of G, or we can look at orthogonal spectra with a G-action. It won't matter for us, we are merely interested in the tt-geometry of $\mathrm{Ho}\,\mathrm{Sp}_G := SH(G)$, the G-equivariant stable homotopy category.

The category $SH(G)$ is nice for us. It is a tt-category, and has a finite set of compact generators: $\{\Sigma^\infty_G G/H_+ \mid H \leq G \text{ subgroup}\}$

Here Σ^∞_G denotes the G-suspension spectrum.

Extension of scalars

As for G-spaces we have for any morphism $f : G \longrightarrow G'$ an induced tt-functor $SH(G') \xrightarrow{f_*} SH(G)$ which for an inclusion $i : H \hookrightarrow G$ gives the restriction res^G_H, which has a lef and right adjoint

$$\begin{tikzcd}
SH(G) \arrow[r, "\mathrm{res}^G_H"] & SH(H) \arrow[l, bend right=60, "\mathrm{Ind}^G_H = G \wedge_H (-)"', "\perp"] \arrow[l, bend left=60, "\mathrm{coind}^G_H = F^G(H,-)", "\perp"']
\end{tikzcd}$$

Since res^G_H is a tt-functor, and it preserves compact objects, we can wonder at what the map $\mathrm{Spc}\, SH(H)^\omega \longrightarrow \mathrm{Spc}\, SH(G)^\omega$ does.

The compact generators $\Sigma^\infty_G G/H_+$ are all tt-rings, i.e. separable commutative ring objects.

This means we can study the internal module category $\Sigma^\infty_G G/H_+ \text{-mod}_{SH(G)}$.

The adjunction $SH(G) \underset{\text{coind}}{\overset{\text{res}}{\rightleftarrows}} SH(H)$ is monadic, which means in particular that

$$SH(H) \simeq \text{Coind}_H^G(\mathbb{1})\text{-}\mathcal{M}od_{SH(G)}$$

$$\text{Coind}_H^G(\mathbb{1}) = \mathcal{M}ap^G(H, \mathbb{S}_G), \quad \mathbb{1} = \mathbb{S}_G = \Sigma_G^\infty G/G_+$$

$$= \Sigma_G^\infty G/H_+$$

Hence we have a tt-equivalence

$$SH(H) \simeq \Sigma_G^\infty G/H_+\text{-}\mathcal{M}od_{SH(G)}$$

As $\Sigma_G^\infty G/H_+$ is compact, this equivalence restricts down to compact objects.

This result is a consequence of a very general theorem

1.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, shift left, "F"] \arrow[dr, shift left, "E"] & \mathcal{D} \arrow[l, shift left, "G"] \arrow[d, phantom, "12"] \\
& G(\mathbb{1})\text{-}\mathcal{M}od_{\mathcal{C}} \arrow[ul, shift left, "R"]
\end{tikzcd}$$

where E and R are extension and restriction of scalars respectively.

Hence, extension of scalars along $\Sigma^\infty_G G/H_+$

$$E : SH(G) \longrightarrow \Sigma^\infty_G G/H_+\text{-}\mathrm{Mod}_{SH(G)}$$

is isomorphic to the restriction $SH(G) \xrightarrow[\mathrm{res}^G_H]{} SH(H)$

This is important because:

1st important thing to remember

Extension of scalars induces a map $\mathrm{Spc}(\Sigma^\infty_G G/H_+\text{-}\mathrm{Mod}_{SH(G)}) \to \mathrm{Spc}\, SH(G)$, whose image is the support of $\Sigma^\infty_G G/H_+$. Hence $\mathrm{Im}(\mathrm{Spc}(\mathrm{res}^G_H)) = \mathrm{Supp}(\Sigma^\infty_G G/H_+)$.

Fixed point functors

The projection $q : G \longrightarrow G/N$ for a normal subgroup N gives the so-called inflation functor

$$\begin{tikzcd} SH(G/N) \arrow[r, "\mathrm{Infl}^G_{G/N}"] \arrow[r, phantom, shift right=3, "\perp"] & SH(G) \arrow[l, bend left=50, "(-)^N"] \end{tikzcd}$$

Its right adjoint is given by "categorical" N-fixed points.

If these fixedpoints are to be nice, we would want

$$\left(\Sigma_G^\infty X\right)^N \cong \Sigma_G^\infty (X)^N,$$

i.e. that fixedpoints commute with constructing G-spectra from G-spaces, but this is not the case... Thus we need to construct other such functors.

Let L_N be the localizing tensor ideal generated by the compact generators containing N, i.e. $L_N = \{ G/K_+ \mid N \not\leq K \}$. We can localize $SH(G)$ at L_N, giving a functor

$$SH(G) \longrightarrow SH(G)/L_N .$$

We can then form the composition:

$$SH(G/N) \xrightarrow{\mathrm{Infl}_{G/N}^{G}} SH(G) \longrightarrow SH(G)/L_N$$

We know $\mathrm{Infl}_{G/N}^{G}$ has an adjoint $(-)^N$.
Can this be extended to the composition?

The N-fixed points of G/K is the empty set when $N \nleq K$ and is the entire set G/K when $N \leq K$. Hence

$$(G/K_+)^N = \begin{cases} \emptyset_+ = * , & N \nleq K \\ G/K_+ , & N \leq K \end{cases}$$

which means that $(-)^N$ kills all the things we have "quotiented out" by anyways

$\Rightarrow$ We have a functor

$$\begin{tikzcd} SH(G/N) \arrow[r] & SH(G) \arrow[r] & \frac{SH(G)}{LN} \arrow[ll, bend left, "(-)^N"] \end{tikzcd}$$

which turns out to be an inverse, i.e. the composition is an equivalence.

This means we have a functor

$$\tilde{\Phi}^N : SH(G) \longrightarrow SH(G)/\mathcal{L}_N \simeq SH(G/N)$$

for any normal subgroup N. This is called the geometric N-fixedpoint functor.

We want a fixedpoint functor for all subgroups $H \leq G$ due to the way $SH(G)$ is generated, i.e. by $\Sigma^\infty_G G/H_+$.

We then define the absolute H-fixed point functor Φ^H by

$$\Phi^H : SH(G) \xrightarrow{res^G_H} SH(H) \xrightarrow{\tilde{\Phi}^H} SH(\{e\}) = SH$$

This is a nice functor. By construction $\tilde{\Phi}^N$ is split by $Infl^G_{G/N}$, so $\tilde{\Phi}^H$ is split by $Infl^H_{\{e\}}$. Extending this to $Infl^G_{\{e\}}$ we get that it splits Φ^H for every H.

The absolute H-fixed points also commute with suspension spectra, i.e.

$$\phi^H(\Sigma^\infty_G X_+) \cong \Sigma^\infty X^H_+ \quad \text{in } SH.$$

Consequences

For any homomorphism $f : G \longrightarrow G'$ s.t. $f(H) = H'$ for a subgroup $H \leq G$ we have

$$\begin{tikzcd}
SH(G') \arrow[r, "res^{G'}_{H'}"] \arrow[d, "f^*"'] \arrow[rrd, bend left=40, "\phi^{H'}"] & SH(H') \arrow[d, "f^*_{|H}"] \arrow[rd, "\tilde{\phi}^{H'}"] & \\
 & & SH \\
SH(G) \arrow[r, "res^G_H"'] \arrow[rru, bend right=40, "\phi^H"'] & SH(H) \arrow[ru, "\tilde{\phi}^H"'] &
\end{tikzcd}$$

commutative up to isomorphism.

We want to turn this into a diagram of Balmer spectra. Can we do so?

We need that all functors are tt-functors and that they preserve compact objects.

Since L_N compactly generated, the localization

$$SH(G) \longrightarrow SH(G/N)$$

is a finite localization, hence preserves compact objects.

All the functors $\mathrm{res}^G_{G'}$, $\mathrm{Infl}^G_{G'}$, $\mathrm{Ind}^G_{G'}$, $\mathrm{coind}^G_{G'}$ are "induced" from maps $G \longrightarrow G'$ and are thus tt-functors. These functors preserve dualizable objects, and we have compact = dualizable in our cases.

Hence res^G_H, f^*, $\tilde{\Phi}^N$ and Φ^H all induce continuous maps on Balmer spectra.

Hence we have a diagram

$$\begin{tikzcd}
 & \mathrm{Spc}\,\mathrm{SH}(H)^\omega \arrow[r] \arrow[dd, "\mathrm{Spc}(f^*_{(H)})"] & \mathrm{Spc}\,\mathrm{SH}(G)^\omega \arrow[dd, "\mathrm{Spc}(f^*)"] \\
\mathrm{Spc}\,\mathrm{SH}^\omega \arrow[ur] \arrow[dr] \arrow[urr, bend left, "\varphi^H"] \arrow[drr, bend right, "\varphi^{H'}"'] & & \\
 & \mathrm{Spc}\,\mathrm{SH}(H')^\omega \arrow[r] & \mathrm{Spc}\,\mathrm{SH}(G')^\omega
\end{tikzcd}$$

In particular: $\mathrm{Spc}(f^*) \circ \varphi^H = \varphi^{H'}$.

Since ϕ^H is split, all these maps φ^H are injective.

2^{nd} main thing to remember

For any subgroup $H \leq G$ we have an injective map $\varphi^H : \mathrm{Spc}\,\mathrm{SH}^\omega \longrightarrow \mathrm{Spc}\,\mathrm{SH}(G)^\omega$ that we can understand from G-sets as their H-fixed points.