Hopkins-Neeman Thm

torsdag 7. oktober 2021 18:49

$S = \bigcup_{I \in \Lambda} V(I)$ Λ set of ideals in R.

Let R be a commutative Noetherian ring.

$$\left\{ \begin{array}{l} \text{Thick subcats.} \\ \text{of } D(R)^c \end{array} \right\} \underset{\tau}{\overset{\sigma}{\rightleftarrows}} \left\{ \begin{array}{l} \text{Spec. closed} \\ \text{subsets} \\ \text{of } \operatorname{Spec}(R) \end{array} \right\}$$

$$\sigma(L) = \{ \mathfrak{p} \in \operatorname{Spec}(R) \mid \exists X \in L \ \operatorname{Tor}_R(X, k(\mathfrak{p})) \neq 0 \}$$

$$\tau(S) = \{ X \in D(R)^c \mid \sigma(X) \subseteq S \}$$

σ og τ gives a bijection.

<u>Plan</u>

① Explain thm. What are the objects

- $D(R)^c$, - $k(\mathfrak{p})$, - σ, τ are well-defined
- Spec. closed

② σ is classifying

③ Lemmata

④ Proof of Hopkins-Neeman

① - $D(R)^c = \{ X \in D(R) \mid X \text{ is compact} \}$

Firstly: $D(R)^c$ is tt, w/ $\operatorname{Tor}_R = \otimes_R^{\mathbb{L}}$.

Also we think tt-subcat is a tt-ideal.

- $k(\mathfrak{p}) \simeq (R)_{\mathfrak{p}} /$ [illegible] is [illegible] field

$(\mathfrak{p}R)_\mathfrak{p}$ is a field

Lemma:

$$X \otimes^{\mathbb{L}}_R k(\mathfrak{p}) \simeq \bigoplus_{n\in\mathbb{Z}} k(\mathfrak{p})[n]^{m_n}$$

Pf

As chain complexes, $\otimes k(\mathfrak{p})$ is v.s.

$H^*(X \otimes k(\mathfrak{p}))$ has a hom. basis $\{\alpha_i \mid i\in I\}$.

Projection

$$X \otimes k(\mathfrak{p}) \longrightarrow \bigoplus_{i:I} k(\mathfrak{p})[|\alpha_i|]$$

Is an isomorphism in homology. $\square$

Consequence:

$$k(\mathfrak{p}) \otimes^{\mathbb{L}}_R M = 0 \iff X \otimes^{\mathbb{L}}_R k(\mathfrak{p}) \otimes M = 0$$

- σ, τ is well-defined.

$$\sigma(L) = \bigcup_{X\in L} \{\mathfrak{p} \mid X \otimes^{\mathbb{L}}_R k(\mathfrak{p}) \neq 0\}$$

$$= \bigcup_{X\in L} \{\mathfrak{p} \mid X_\mathfrak{p} \neq 0\}$$

$$= \bigcup_{X\in L} \{\mathfrak{p} \mid H^*(X)_\mathfrak{p} \neq 0\}$$

$$= \bigcup_{X\in L} \{\mathfrak{p} \mid \mathrm{Ann}_R(H^*(X)) \subseteq \mathfrak{p}\}$$

$$= \bigcup_{X\in L} V(\mathrm{Ann}_R(H^*(X)))$$

Assume σ is a support data.

$$\underbrace{\tau(S)}_{\text{subcat.}} = \{X \in D(R)^c \mid \sigma(X) \subseteq S\}$$

- $\sigma(X) = \sigma(X[n])$

- $A \to B \to C \to A[1] \Rightarrow \sigma(C) \subseteq \sigma(A \oplus B)$
- $\sigma(A) \cup \sigma(B) \subseteq \sigma(A \oplus B) \subseteq S$
  $\Rightarrow \sigma(A) \subseteq S, \quad \sigma(B) \subseteq S.$

②. <u>σ defines a support data</u>

<u>Pf</u> Most of these are by def.

Show case $\sigma(A \otimes_R^{\mathbb{L}} B) = \sigma(A) \cap \sigma(B)$.

$\subseteq)$ $(\sigma(A) \cap \sigma(B))^c$

$= \{p \mid A \otimes_R^{\mathbb{L}} k(p) = 0\} \cup \{p \mid B \otimes_R^{\mathbb{L}} k(p) = 0\}$

$\subseteq \{p \mid A \otimes_R^{\mathbb{L}} B \otimes_R^{\mathbb{L}} k(p) = 0\} = \sigma(A \otimes_R^{\mathbb{L}} B)$

$\supseteq)$ $\sigma(A) \cap \sigma(B) =$

$\{p \mid A \otimes_R^{\mathbb{L}} k(p) \neq 0\} \cap \{p \mid B \otimes_R^{\mathbb{L}} k(p) \neq 0\}$

$\subseteq \{p \mid A \otimes_R^{\mathbb{L}} B \otimes_R^{\mathbb{L}} k(p) = 0\} = \sigma(A \otimes_R^{\mathbb{L}} B)$

Lemma: If $A \otimes_R^{\mathbb{L}} B \otimes_R^{\mathbb{L}} k(p) = 0$ $\Rightarrow A \otimes_R^{\mathbb{L}} k(p) = 0$ or $B \otimes_R^{\mathbb{L}} k(p) = 0$

Use here

<u>σ is classifying</u>

Spec(R) is spectral.

Thomasons are spec. closed.

Every object in $D(R)^c$ is compact

$\Rightarrow$ Every object in $D(R)^c$ is dualizable

$\Rightarrow$ Every tt-ideal is radical!

③ ~~Lemma~~

Let $p \in \operatorname{Spec}(R)$, then there exists a complex $R/\!/p$, s.t. $\sigma(R/\!/p) = \bar{p}$

Johanne's master thesis, she shows that the koszul complex $R/\!/p = R/\!/(a_1, \ldots, a_n)$

Thm

For ever thick subcat $\mathcal{E}$ of $D(R)^c$, there is a functor L_f^E, called finite localization functor. This functor is smashing, i.e. $L_f^E(X) \simeq L_f^E(R) \otimes X$

L_f^E has kernel E, i.e. $L_f^E(X) \simeq 0 \Leftrightarrow X \in E$

This thm is 3.3.3. in Axiomatic stable htpy theory.

Lemma:

The residue fields detect zero.

I.e. $M \simeq 0 \Leftrightarrow M \otimes_R^L k(p) \simeq 0$.

(4) Proof of Hopkins-Neeman thm.

- Show that σ is epi.

Let S be spec. closed in $\mathrm{Spec}(R)$.

Let $p \in S \Rightarrow \bar{p} \subseteq S \Rightarrow \sigma(R/\!/p) \subseteq S$

p was arbitrary $\Rightarrow R/\!/p$ exists for any p.

- Show that σ is split mono.

$\tau \circ \sigma \simeq id_{thick}$

$L \rightsquigarrow \tau \circ \sigma(L) = \{X \in D(R)^c \mid \sigma(X) \subseteq \sigma(L)\}$

Pick $X \in \tau \circ \sigma(L)$. Then $\sigma(X) \subseteq \sigma(L)$.

$\Leftrightarrow X \in L$

$$X \in \mathcal{L} \iff L^L_J(X) \simeq 0 \iff L^L_J(R) \otimes^L_R X \simeq 0$$

$$\iff \forall p \in \mathrm{Spec}(R) \quad k(p) \otimes^L_R L^L_J(R) \otimes^L_R X \simeq 0$$

$\boxed{\mathcal{L} \subseteq \tau \circ \sigma(\mathcal{L}),}$ $\sigma(\mathcal{L}) = \bigcup_{X \in \mathcal{L}} \sigma(X)$

$\Rightarrow X \in \mathcal{L} \Rightarrow \sigma(X) \subseteq \sigma(\mathcal{L})$ ok.

$\boxed{\mathcal{L} \supseteq \tau \circ \sigma(\mathcal{L})}$

Assume $X \in \tau \circ \sigma(\mathcal{L})$, then $\sigma(X) \subseteq \sigma(\mathcal{L})$.

$X \otimes^L_R k(p) \not\simeq 0$ for $p \in \sigma(X) \subseteq \sigma(\mathcal{L})$. $Y \in \mathcal{L}$

$Y \otimes^L_R k(p) \not\simeq 0$. $L^L_J(Y) \simeq L^L_J(R) \otimes Y \simeq 0$

$$(k(p) \otimes^L_R Y) \otimes^L_R (X \otimes^L_R L^L_J(R)) \simeq 0$$

$\Rightarrow k(p) \otimes^L_R X \otimes^L_R L^L_J(R) \simeq 0 \ \forall p \subseteq R \iff X \in \mathcal{L}$ $\square$