TT-geometry: The comparison map

Rc. 2010, all homeom. between $\mathrm{Spc}\,\mathcal{K}$ and "more explicit" top. spaces came from the final map of $\mathrm{Spc}\,\mathcal{K}$, i.e. $X \to \mathrm{Spc}\,\mathcal{K}$. Today we explore some maps in the other direction into spectral spaces (i.e. homeo to spectrum of a ring.)

So to compare $\mathrm{Spc}\,\mathcal{K}$ with $\mathrm{Spec}\,R$ for a ring R, what should R be?

Definition

For $\mathcal{K}$ ess. small tt-cat., the central ring is $R_{\mathcal{K}} = \mathrm{End}_{\mathcal{K}}(\mathbb{1})$ for $\mathbb{1}$ the monoidal unit.

Note:
$R_{\mathcal{K}}$ is commutative since the product (composition) coincides with $\otimes$:

$$\begin{tikzcd}
1 \arrow[r, "\cong"] \arrow[rrr, bend left, "f\otimes g"] \arrow[d, "g"'] & 1\otimes 1 \arrow[r, "f\otimes g"] \arrow[dr, "\mathrm{id}\otimes g"] & 1\otimes 1 \arrow[r, "\cong"] & 1 \\
1 \arrow[rr, equal] \arrow[r, "\cong"] & 1\otimes 1 \arrow[u, "f\otimes\mathrm{id}"] \arrow[r, "\cong"] & 1 \arrow[ur, "f"'] &
\end{tikzcd} \quad : f\otimes g = f\circ g$$

This makes $\mathrm{Hom}_{\mathcal{K}}(a,b)$ into a $R_{\mathcal{K}}$-module via $f\cdot g := f\otimes g$ for $f: \mathbb{1}\to\mathbb{1}$ and $g: a\to b$ (If $g: a\to\mathbb{1}$ then $f\cdot g = f\circ g$). and composition becomes $R_{\mathcal{K}}$-bilinear, $h\circ(f\cdot g_1 + g_2) = f\cdot(h\circ g_1) + g_2$. The left and right module structure coincides.

Another choice could be the graded version:

Definition

The graded central ring is $R^{\bullet}_{\mathcal{K}} = \mathrm{Hom}^{\bullet}_{\mathcal{K}}(\mathbb{1},\mathbb{1})$ where $\mathrm{Hom}^{i}_{\mathcal{K}}(\mathbb{1},\mathbb{1}) = \mathrm{Hom}(\mathbb{1},\Sigma^i\mathbb{1})$

This has a composition $(b \xrightarrow{g} \Sigma^j c,\ a \xrightarrow{f} \Sigma^i b) \longmapsto \left(\begin{tikzcd} a \arrow[r, "g\circ"] \arrow[d, "f"'] & \Sigma^{i+j} c \\ \Sigma^i b \arrow[ur, "\mathrm{id}^{\otimes i}\otimes g"'] & \end{tikzcd}\right)$ making it unital assoc. skew-commutative ring, and $\mathrm{Hom}^{\bullet}_{\mathcal{K}}(a,b)$ becomes graded left and right modules, although they don't coincide. (unless $a=b=\mathbb{1}$)

For invertible $u\in\mathcal{K}$, one could similarly define $\mathrm{Hom}_{\mathcal{K}}(\mathbb{1},\Sigma^i u)$ for $R^{\bullet}_{\mathcal{K},u}$, and thus we have a family of spaces $\mathrm{Spc}(R^{\bullet}_{\mathcal{K},u})$ we could compare with. ($R^{\bullet}_{\mathcal{K},u}$ is not always skew commutative).

$(*)$ $\mathrm{Hom}^{\bullet}_{\mathcal{K}}(a,b) = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}^{i}_{\mathcal{K}}(a,b) = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{K}}(a,\Sigma^i b)$

Def: $R^{hom} = \bigcup_{i\in\mathbb{Z}} R^i$, i.e. homogeneous elements of R, and
$R^{even} := \bigcup_{i\in\mathbb{Z}} R^{2i}$.

Since $R^\circ_{\mathcal{K}}$ is skew-commu, $R^{even}_{\mathcal{K}}$ will be the center of $R^\circ_{\mathcal{K}}$ (where everything commutes).

Definition
The homogeneous spectrum of a graded ring $\mathrm{Spec}^h(R^\circ)$ consists of homogeneous ideals p° of R° which are prime. This has a Zariski top. like other rings. (As Drew says, just use "homogeneous" enough, and everything should work out).

Definition
For R° skew-commutative, $S \subset R^{even}$, we can localise at S.
$(S^{-1}R^\circ)^i = \{\frac{f}{s} \mid f \in R^{i+j}, s \in S \cap R^j, j \in \mathbb{Z}\}$. For more general subsets $T \subset R^{hom}$, take $S = \{t^2 \mid t \in T\}$. This supplies us with a localisation map $R^\circ \to S^{-1}R^\circ$ which induces a map $\mathrm{Spec}^h(S^{-1}R^\circ) \hookrightarrow \mathrm{Spec}^h(R^\circ)$.
Choosing p° prime hom. ideal, and $S_{p^\circ} = \{s \in R^{hom} \mid s \notin p^\circ\}$, $S^{-1}_{p^\circ}R^\circ = R^\circ_{p^\circ}$ so it is a graded local ring, and $\mathrm{Spec}^h(R^\circ_{p^\circ}) \to \mathrm{Spec}^h(R^\circ)$
unique max. ideal $\mapsto p^\circ$.

Construction
For tt-cat $\mathcal{K}$, suppose $S \subset R^{even}_{\mathcal{K}}$ multiplicative subset. Then assoc. to $S^{-1}R_{\mathcal{K}}$ we have modules $S^{-1}\mathrm{Hom}^\circ_{\mathcal{K}}(a,b)$. This defines a cat. $S^{-1}\mathcal{K}$, obj. from $\mathcal{K}$ and $\mathrm{Hom}_{S^{-1}\mathcal{K}}(a,b) = (S^{-1}\mathrm{Hom}^\circ_{\mathcal{K}}(a,b))^\circ$ with nat. functor $q_S : \mathcal{K} \to S^{-1}\mathcal{K}$.

Theorem
For $\mathcal{J} = \langle \mathrm{cone}(s) \mid s \in S \rangle$ (this is thick triang. ⊗-ideal) with Verdier localisation $q : \mathcal{K} \to \mathcal{K}/\mathcal{J}$, there is eq. $\alpha : S^{-1}\mathcal{K} \to \mathcal{K}/\mathcal{J}$ s.t. $\alpha \circ q_S = q$. In particular $S^{-1}\mathcal{K}$ is a tt-cat. s.t. q_S is morphism of tt-cats.

SHOULD COME LAST

Now, since we are in triang. cat., a nice way to associate obj. of $\mathcal{K}$ with elements of $R^{\bullet}_{\mathcal{K}}$, is through cones of morphisms $f: \mathbb{1} \to \Sigma^{i} \mathbb{1}$.

Definition
$\mathcal{P} \in \operatorname{Spc} \mathcal{K}$ prime, define $\rho^{\bullet}_{\mathcal{K}} : \operatorname{Spc} \mathcal{K} \to (R^{\bullet}_{\mathcal{K}}\text{-mod})$

$$\mathcal{P} \mapsto \{ f \in R^{\bullet\, \text{hom}}_{\mathcal{K}} \mid \operatorname{cone} f \notin \mathcal{P} \}.$$

Warning: $\rho^{\bullet}_{\mathcal{K}}$ reverses inclusions!

Thm. 5.3 (reduced)

a) $\rho^{\bullet}_{\mathcal{K}}(\mathcal{P})$ is hom. prime ideal.

b) $\rho^{\bullet}_{\mathcal{K}} : \operatorname{Spc} \mathcal{K} \to \operatorname{Spec}^{h} R^{\bullet}_{\mathcal{K}}$ is continuous.

c) $\rho^{\bullet}$ defines nat. transf. between $\mathcal{K} \mapsto \operatorname{Spc} \mathcal{K}$ and $\mathcal{K} \mapsto \operatorname{Spec}^{h}(R^{\bullet}_{\mathcal{K}})$.

Corollary
Similarly holds for $\rho_{\mathcal{K}} = (-)^{\circ} \circ \rho^{\bullet}_{\mathcal{K}}$ ($\operatorname{Spc} \mathcal{K} \to \operatorname{Spec} R_{\mathcal{K}}$), i.e.: $\rho_{\mathcal{K}}(\mathcal{P}) = \{ f \in R_{\mathcal{K}} \mid \operatorname{cone} f \notin \mathcal{P} \}$

Theorem 7.3 (finitely presented with fin. presented modules when fin. gen.)
$\mathcal{K}$ tt-cat s.t. $R^{\bullet}_{\mathcal{K}}$ is coherent (e.g. noetherian) then $\rho^{\bullet}_{\mathcal{K}} : \operatorname{Spc} \mathcal{K} \to \operatorname{Spec}^{h} R^{\bullet}_{\mathcal{K}}$ is surjective.
and $\rho_{\mathcal{K}} : \operatorname{Spc} \mathcal{K} \to \operatorname{Spec} R_{\mathcal{K}}$

Thm 7.12

$\mathcal{K}$ connective tt-cat, i.e. $\operatorname{Hom}_{\mathcal{K}}(\mathbb{1}, \Sigma^{i} \mathbb{1}) = 0 \ \forall i > 0$, then $\rho_{\mathcal{K}} : \operatorname{Spc} \mathcal{K} \to \operatorname{Spec} R_{\mathcal{K}}$ is surj.

Corollary
$\mathcal{K}$ connective with $R^{<0}_{\mathcal{K}} = \bigoplus_{i<0} \operatorname{Hom}(\mathbb{1}, \Sigma^{i} \mathbb{1})$ a nil-ideal then $\operatorname{Spec}^{h}(R^{\bullet}_{\mathcal{K}}) \cong \operatorname{Spec} R_{\mathcal{K}}$ via $(-)^{\circ}$ and $\rho^{\bullet}_{\mathcal{K}}$ is surjective.

With these relatively ~~all~~ easy surjectivity conditions, one might imagine getting a cover of $\operatorname{Spc} \mathcal{K}$ from pulling

$\operatorname{Spc} R_{\mathcal{K}}$ back before even knowing anything about $\operatorname{Spc}\mathcal{K}$.
But does these maps intertwine nicely with the theory presented thus far?[2]

Last week, Thomas showed that $f: \operatorname{Spec} R \xrightarrow{\simeq} \operatorname{Spc} D(R)^c$ by applying the universality of $(\operatorname{Spc}\mathcal{K}, \operatorname{supp})$ ~~and then~~ and the Hopkins-Neeman thm which equips Spec R with support datum $\sigma(M_\bullet) = \{p \in \operatorname{Spec}(R) \mid k(p)\otimes M_\bullet \neq 0\} = \{p \in \operatorname{Spec}(R) \mid (M_\bullet)_p \neq 0 \text{ in } D(R_p)^c\}$.
And then associated map becomes

$$f(p) = P(p) = \{M_\bullet \in D(R)^c \mid (M_\bullet)_p = 0 \text{ in } D(R_p)^c\}.$$

It turns out, that $\rho_{\mathcal{K}}$ becomes the inverse map.

$$\operatorname{Spec} R \xrightarrow{f} \operatorname{Spc} D(R)^c \to \operatorname{Spec} R_{\mathcal{K}} \cong \operatorname{Spec} R$$

$$R_{\mathcal{K}} = \operatorname{End}_{D(R)^c}(R_\bullet, R_\bullet) \cong R^{op} \overset{\text{commutative}}{=} R$$

We show $\rho_{\mathcal{K}}(P(p)) = p$:
For $f \in \rho_{\mathcal{K}}(P(p))$, $f: 1 \to 1$, by definition cone $f \notin P(p)$ so examining the triangle $1_p \xrightarrow{f} 1_p \to \operatorname{cone} f_p \to \Sigma 1_p$ in $D(R_p)^c$, if $\operatorname{cone} f_p \neq 0$ then f is not invertible in $D(R_p)^c$ and if f is not invertible after localising at p, then $f \in p$ (recall R_p inverts everything outside of p) "□"

In general, one could wonder if we get surjectivity of $\rho_{\mathcal{K}}$ for free, what about injectivity? one can prove injectivity of $\rho_{\mathcal{K}}: \operatorname{Spc} R \to \operatorname{Spc} D(R)^c$ directly without Hopkins-Neeman in more than one way, but this I did not have time to prepare. A proof is presented in Section 8 of (Spectra)[3].