TDA – The isometry thm

2021

(Top. persistence in geom. and anal.)

Recall some definitions, updated

Def Persistence module

(V, π), V_t fin-dim vec. space $\forall t \in \mathbb{R}$

$\pi_{s,t} : V_s \to V_t$ $\forall s, t \in \mathbb{R}$ such that

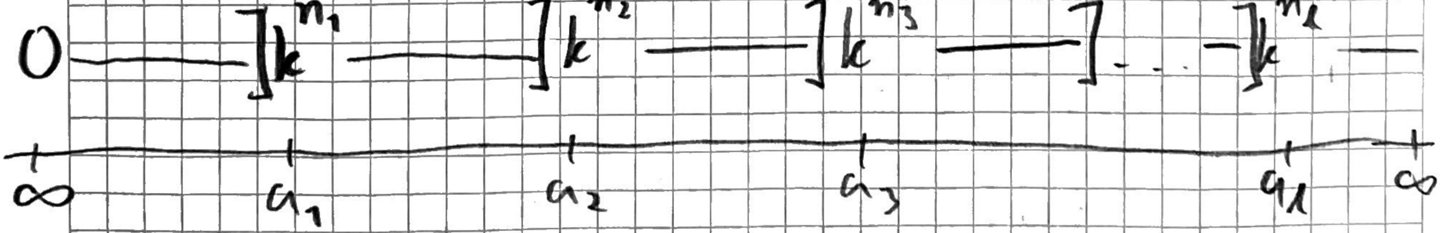
1)
$$\begin{tikzcd} V_s \arrow[rr, "\pi_{s,r}"] \arrow[dr, "\pi_{s,t}"'] & \circlearrowleft & V_r \\ & V_t \arrow[ur, "\pi_{t,r}"'] & \end{tikzcd}$$

2) $\forall t \in \mathbb{R} \setminus \{\text{fin \# pts}\}$, $\exists\ U \ni t$ open where $\pi_{r,s}$ is iso $\forall\ r, s \in U$

3) $\forall t \in \mathbb{R}$, $\forall\ s \leq t$ suff. close to t, $\pi_{s,t}$ is an iso

4) $\exists\ S$ such that $V_s = 0$ $\forall\ s \leq S$

Picture:



Morphisms: $f : (V, \pi) \to (W, \tau)$,
$$\begin{tikzcd} V_s \arrow[r, "\pi_{s,t}"] \arrow[d, "f_s"'] & V_t \arrow[d, "f_t"] \\ W_s \arrow[r, "\tau_{s,t}"'] & W_t \end{tikzcd}$$

Note: These are the objects we get when taking $H_k(VR(X))$, for some k.

Shift: $V[\delta]$, $V[\delta]_t := V_{t+\delta}$, π likewise

$\phi^\delta_V : V \to V[\delta]$

$\phi^\delta_{V,t} = \pi_{t,t+\delta}$

<u>Interleaving</u>

(V,π), (W,τ)

δ-interleaved if $\exists$ morphisms $F: V \to W[\delta]$, $G: W \to V[\delta]$ such that

$$\begin{tikzcd} V \arrow[rr, "\phi^{2\delta}_V"] \arrow[dr, "F"'] & \arrow[d, phantom, "\circlearrowleft"] & V[2\delta] \\ & W[\delta] \arrow[ur, "G[\delta]"'] & \end{tikzcd} \qquad \begin{tikzcd} W \arrow[rr, "\phi^{2\delta}_W"] \arrow[dr, "G"'] & \arrow[d, phantom, "\circlearrowleft"] & W[2\delta] \\ & V[\delta] \arrow[ur, "F[\delta]"'] & \end{tikzcd}$$

<u>Example from last time</u>

Let $f,g: \mathbb{R} \to \mathbb{R}$, $\|f-g\|_\infty \le \varepsilon$, then

$d_I(\mathbb{F}, \mathbb{G}) \le \varepsilon$, where

$\mathbb{F}_t = H_0(f^{-1}(-\infty,t))$

$\pi_{s,t}: H_0(f^{-1}(-\infty,s)) \longrightarrow H_0(f^{-1}(-\infty,t))$

from homology, $f^{-1}(-\infty,s) \subseteq f^{-1}(-\infty,t)$

And similarly for $\mathbb{G}$. Assume changes at fin. many pts.

Sol: Want $f: \mathbb{F} \to \mathbb{G}[\varepsilon]$. Take $t \in \mathbb{R}$,

want $f_t : H_0(f^{-1}(-\infty, t)) \to H_0(g^{-1}(-\infty, t+\varepsilon))$

But ~~$g(x) \leq t+\varepsilon \iff f(x) \leq t$~~

when $f(x) < t$, $g(x) < t+\varepsilon$, so

$$f^{-1}(-\infty, t) \subseteq g^{-1}(-\infty, t+\varepsilon),$$ so get

map in homology from inclusion

$$\begin{tikzcd}
H_0(f^{-1}(-\infty,t)) \arrow[rr, "\pi_{t,t+2\varepsilon}"] \arrow[dr, "f_t"'] & \circlearrowleft & H_0(f^{-1}(-\infty,t+2\varepsilon)) \\
& H_0(g^{-1}(-\infty,t+\varepsilon)) \arrow[ur, "g_{t+\varepsilon}"'] &
\end{tikzcd}$$

Commutes since homology functor.

<u>Good</u> <u>Isometry thm</u>

The map $V \mapsto B(V)$ that sends a pers. module to a barcode is an isometry, i.e.

$$d_I(V, W) = d_B(B(V), B(W))$$

Useful: <u>Interval modules</u>

Given $I = [a, b]$, $a \in \mathbb{R}$, $b \in \mathbb{R} \cup \{\infty\}$, get persistence mod

$$k(I)_t = \begin{cases} k & t \in [a,b] \\ 0 & \text{ow} \end{cases} \qquad \pi_{s,t} = \begin{cases} id & s,t \in [a,b] \\ 0 & \text{ow} \end{cases}$$

The Normal Form theorem

Given a persistence module $(\mathbb{V}, \pi)$, $\exists$ intervals $\{I_i\}_{i=1}^{N}$, $I_i = (a_i, b_i]$, b_i can be ∞, $I_i \neq I_j$ $i \neq j$.

such that

$$\mathbb{V} \simeq \bigoplus_{i=1}^{N} k(I_i)^{n_i} \qquad n_i \in \mathbb{N}$$

Essentially barcodes ($\mathbb{B}(V)$ is $\{(I_i, n_i)\}_{i=1}^{N}$)

Prop (one dir. of isom. thm)

$$d_I(\mathbb{V}, \mathbb{W}) \leq d_B(\mathbb{B}(V), \mathbb{B}(\mathbb{W}))$$

Proof

$d_B(\mathbb{B}(V), \mathbb{B}(\mathbb{W})) = \delta$, want δ-interleaving of $\mathbb{V}$ and $\mathbb{W}$. Normal form:

$$\mathbb{V} = \bigoplus_{I \in \mathbb{B}(V)} k(I) \qquad \mathbb{W} = \bigoplus_{J \in \mathbb{B}(W)} k(J)$$

We have a δ-matching $\mu : \mathbb{B}(V) \longrightarrow \mathbb{B}(\mathbb{W})$, by assump.

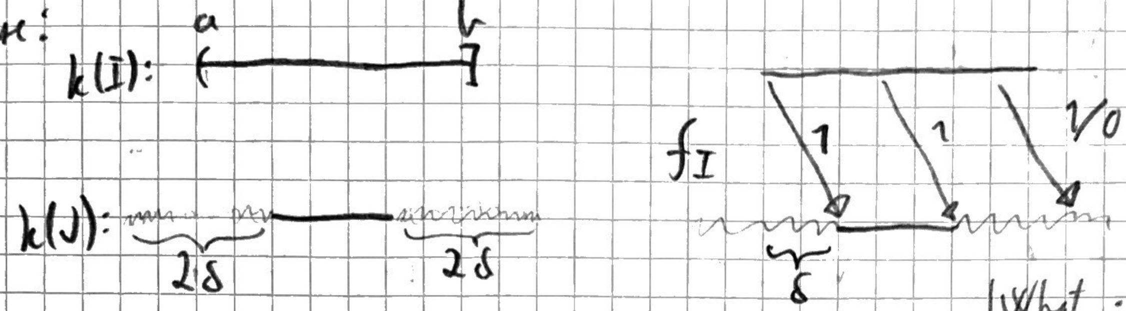
Set

$$\mathbb{V}_M = \bigoplus_{I \in \operatorname{coim} \mu} k(I) \qquad \mathbb{W}_M = \bigoplus_{J \in \operatorname{im} \mu} k(I)$$

$$\mathbb{V}_U = \mathbb{V} / \mathbb{V}_M \qquad \mathbb{W}_U = \mathbb{W} / \mathbb{W}_M$$

Take first some $I \in \operatorname{coim} \mu$, $\mu(I) = J$.

Picture:

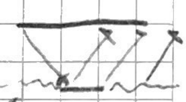


(What if different?)

This is a map $k(I) \to k(J)[\delta]$ no matter what part of ~~~ exists.

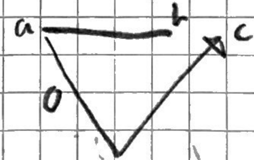
Likewise, get $g_J : k(J) \to k(I)[\delta]$

$g_J \circ f_J = \Phi^{2\delta}_{k(I)}$ since Φ is 1 when domain and codomain are k, and 0 o/w.



So get δ-interleaving $f_M : V_M \to W_M[\delta]$, $g_M : W_M \to V_M[\delta]$.

Now for intervals $I \in V_U$, we can just set $f_I = 0$, since their length is less than 2δ (μ is δ-matching).



$\pi_{a,c} = 0$, so no problem

So these $k(I)$ are δ-interleaved with 0.

Same with $J \in V_U$, $g_J = 0$.

This gives δ-interleaving

$f|_{V_M} = f_M$, $f|_{V_U} = 0$ $\quad (g|_{W_M} = g_M,\ g|_{W_U} = 0)$

$\square$

The converse

Long, but not hard. $V \xrightarrow{f} W[\delta] \xrightarrow{g[\delta]} V[2\delta]$ $\mu: B(V) \to B(W)$

Sketch: We want to go from interleaving to matching.

For barcode B, interval $I = (b, c]$, set

$B_I^- \subseteq B$ to be all intervals of the form

$(a, c]$, $a \leq b$ (start before a, end at c).

Fact

Assume we have an injection $i: (V, \pi) \to (W, \tau)$.

Then $|B(V)_I^-| \leq |B(W)_I^-|$ $\forall$ possible I. (Show!)

So we get the following induced matching

$\mu_i : B(V) \to B(W)$:

For every $d \in \mathbb{R} \cup \{\infty\}$, sort the bars $(-, d] \in B(V)$

as $(b_1, d] \supseteq (b_2, d] \supseteq \dots \supseteq (b_k, d]$ $b_1 \leq \dots \leq b_k$

Sim. for $(-, d] \in B(W)$,

$(c_1, d] \supseteq \dots (c_l, d]$ $c_1 \leq \dots \leq c_l$

Match $(b_i, d]$ to $(c_i, d]$.

Because of fact, all intervals in $B(V)$ are matched. Also easy to see $\mu_i((b, d]) = (c, d]$, $c \leq b$, $\forall I$.

Do opposite with surjection, also get matching.

Now we can factor $f : V \to W[\delta]$ as

$$f : V \xrightarrow{r} \mathrm{Im} f \xhookrightarrow{i} W[\delta],$$ and get matchings μ_i, μ_r.

$$\mu_i \circ \mu_r : B(V) \to B(W[\delta]),$$ add

$$\Psi_\delta : B(W[\delta]) \to B(W)$$
$$[a,b] \longmapsto (a+\delta, b+\delta]$$

to get matching $B(V) \to B(W)$.

Ex